Math 105 — Second Midterm — April 14, 2021

EXAM SOLUTIONS

---

1. This exam has 7 pages including this cover. There are 6 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer mathematical questions about exam problems during the exam.
3. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Partial credit will be awarded for correct work.
4. Problems may ask for answers in *exact form*. Recall that $x = \frac{1}{3}$ is an exact answer to the equation $3x = 1$, but $x = 0.333$ is not.
5. You do not need to "simplify" your answers unless asked to do so.
6. You must use the methods learned in this course to solve all problems. Logarithm functions taught in this course include "log" (log base 10) and "ln" (natural log).
7. You may use one pre-written page of notes, on an 8.5"x11" standard sheet of paper, with whatever you want written on both sides.
8. You will not be allowed to use any other resources, including calculators, other notes, or the book.
9. You must write your work and answers on **blank, white, physical paper**.
10. You must write your **initials and UMID**, but not your name or uniqname, in the upper right corner of every page of work. Make sure that it is visible in all scans or images you submit.
11. Make sure that all pages of work have the relevant problem number clearly identified.

Problem	Points
1	2
2	10
3	14
4	11
5	12
6	11

Total	60

1. [2 points] **There <u>is</u> work to submit for this problem. Read it carefully.**

- You may use your one pre-written page of notes, on an 8.5" by 11" standard sheet of paper, with whatever you want written on both sides.
- You are <u>not</u> allowed to use any other resources, including calculators, other notes, or the book.
- You may <u>not</u> use any electronic device or the internet, except to access the Zoom meeting for the exam, to access the exam file itself, to submit your work, or to report technological problems via the Google forms we will provide to do so. The one exception is that you may use headphones (e.g. for white noise) if you prefer, though please note that you need to be able to hear when the end of the exam is called in the Zoom meeting.
- You may <u>not</u> use help from any other individuals (other students, tutors, online help forums, etc.), and may <u>not</u> communicate with any other person other about the exam until **11 am on Thursday** (Ann Arbor time).
- The one exception to the above policy is that you may contact the proctors in your exam room via the chat in Zoom if needed.
- Violation of any of the policies above may result in a score of zero for the exam, and, depending on the violation, may result in a failing grade in the course.

As your submission for this problem, you must write "I agree," and write your initials and UMID number to signify that you understand and agree to this policy. By doing this you are attesting that you have not violated this policy.

2. [10 points] Below are some values of functions $f(x), g(x)$, and $h(x)$.

x	0	1	2	3	4
$f(x)$	2	0	4	4	3
$g(x)$	4	3	b	1	1
$h(x)$	3	a	3	0	0
$k(x)$	0	2	-3	1	0

Additionally:

- $h(x) = f(g(x))$
- The domain of $f(x)$ is $\{0, 1, 2, 3, 4\}$.
- $k(x)$ is an even, periodic function with period 10.

a. [6 points] Find the following values, or explain why they cannot be found from the given information. Be sure to show your work or explain your reasoning.

(i) a

(ii) b

(iii) $k(18)$

Solution:

(i) According to the table, $a = h(1)$, which by definition is $f(g(1))$. Using the table, $g(1) = 3$, so $f(g(1)) = f(3) = 4$. Therefore, $a = 4$.

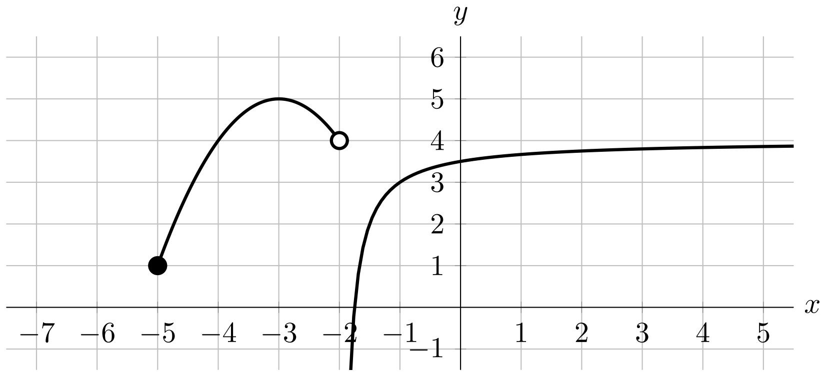
(ii) According to the table, $b = g(2)$. We need to use the other facts from the table to find this value. In particular, we know $h(2) = f(g(2)) = 3$. So $g(2)$ must be a number b so that $f(b) = 3$. The only such value is 4. Therefore, $b = 4$.

(iii) Since $k(x)$ is periodic with period 10, we know that $k(18) = k(8) = k(-2)$. Further, we know $k(x)$ is even, so that $k(-2) = k(2) = -3$. Therefore, $k(18) = -3$.

b. [4 points] Find all solutions to the equation $k(f(x)) = 0$.

Solution: According to the table, $k(u) = 0$ when $u = 0$ or 4, so we need to find x-values so that $f(x) = 0$ or $f(x) = 4$. The table tells us $f(1) = 0$ and $f(2) = f(3) = 4$, so the solutions are $x = 1, 2$, and 3.

3. [14 points] Consider the function $f(x)$, graphed below. Note that $f(x)$ has one vertical asymptote and one horizontal asymptote, and $f(x)$ is not defined for x values to the left of those shown in the graph.



a. [7 points] Find:

(i) the domain of $f(x)$

(ii) the range of $f(x)$

(iii) an equation for the horizontal asymptote of $f(x)$

(iv) an equation for the vertical asymptote of $f(x)$

Solution:

(i) $[-5,-2) \cup (-2,\infty)$

(ii) $(-\infty, 5]$

(iii) $y = 4$

(iv) $x = -2$

b. [7 points] Let $g(x) = 3f(-4(x-2)) + 1$. Find the following. Show how you obtained your answers, either by showing work, drawing diagrams, or explaining your reasoning.

(i) the domain of $g(x)$

(ii) an equation for the horizontal asymptote of $g(x)$

(iii) an equation for the vertical asymptote of $g(x)$

Solution: The graph of $g(x)$ can be found from the graph of $f(x)$ using the following transformations:

- a horizontal contraction by a factor of $1/4$
- a reflection across the y-axis
- a horizontal shift 2 to the right
- a vertical stretch by a factor of 3
- a vertical shift up by 1.

There are several different orders in which these transformations can be applied, but the stretch/compress and reflection in each direction must be applied before the corresponding shift.

(i) We can look at how the transformation affects each of endpoints in the domain from part **a.** The domain is only affected by the first three transformations.

- First, compress $[-5,-2)\cup(-2,\infty)$ by a factor of $1/4$: this gives us $[-5/4,-1/2)\cup(-1/2,\infty)$
- Then reflect across the y-axis. This sends each point (x, y) to the point $(-x, y)$. Note that this means we must reverse the order in which the endpoints show up in the interval: $(-\infty, 1/2)\cup(1/2, 5/4]$.
- Finally, shift 2 to the left: $(-\infty, 5/2)\cup(5/2, 13/4]$.

This gives a final answer of $(-\infty, 5/2)\cup(5/2, 13/4]$.
Another way to find these solutions would be to set $-5 \le -4(x-2) < -2$ and solve for x. Again, it is important to remember that multiplying by -1 will reverse the order of the inequalities, and that $+\infty$ will be transformed to $-\infty$.

(ii) The horizontal asymptote to $f(x)$ is the line $y = 4$. Since this line corresponds to the variable on the vertical axis, it is affected by the vertical transformations. We first multiply by 3 and then add 1, giving $y = 13$.

(iii) The vertical asymptote $x = -2$ for $f(x)$ will be transformed by the horizontal transformations. Multiplying by $-1/4$ and then adding 2 gives $x = 5/2$. Note that we can also see this in our answer for the domain of $g(x)$, where $5/2$ was not included in the domain.

4. [11 points] Mia and Jonathan sell vegetables at the farmer's market at different booths. Their revenues, in **hundreds** of dollars, h hours after 9 am on a particular day are $M(h)$ (for Mia's revenue) and $J(h)$ (for Jonathan's revenue). Assume that the two functions are invertible.

a. [2 points] Give a practical interpretation of the equation $J(2) = 3$.

Solution: This means that Jonathan's revenue at 11 am is equal to \$300.

b. [3 points] Give a practical interpretation of the expression $J(M^{-1}(4))$, or explain why the expression does not make sense in the context of the problem.

Solution: $J(M^{-1}(4))$ is Jonathan's revenue, in hundreds of dollars, at the time when Mia's revenue is \$400.

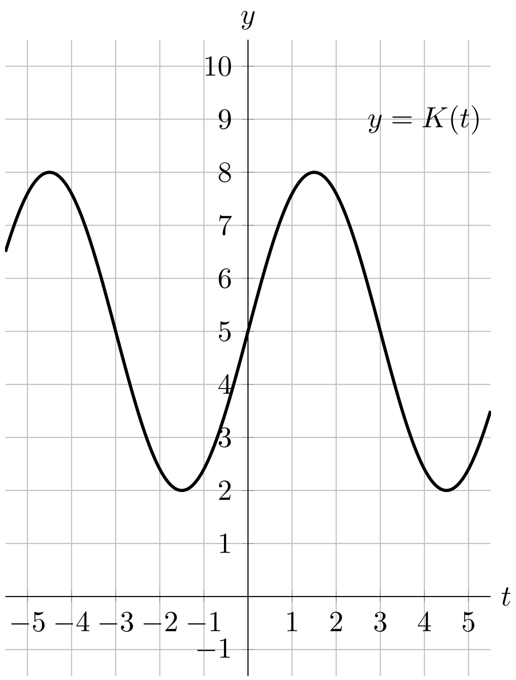
c. [3 points] Write an equation corresponding to the following statement: Mia's revenue at 12pm is \$100 less than twice Jonathan's revenue at 11 am.

Solution: $M(3) = 2J(2) - 1$.

d. [3 points] Let $T(k)$ be the total revenue, in **dollars** of both Mia and Jonathan k **minutes** after 9 am. Find a formula for $T(k)$ in terms of M and/or J.

Solution: $T(k) = 100(M(k/60) + J(k/60))$.

5. [12 points] The graph of a sinusoidal function $y = K(t)$ is given below.



a. [7 points] Find the following.

(i) The amplitude of $K(t)$.
(ii) The midline of $K(t)$.
(iii) The period of $K(t)$.
(iv) A formula for $K(t)$.

Solution:

(i) $(8 - 2)/2 = 3$

(ii) $y = 5$

(iii) 6

(iv) Note that the function starts at its midline at $t = 0$ and increases to the right, so we can use $\sin(t)$ without a horizontal shift or a reflections. Using the values we've found above, we get $3\sin((2\pi/6)t) + 5$.

b. [5 points] Find the first **three** positive values of t for which $K(t) = 7$. Give your answer in exact form.

Solution: We want to know when

$$3\sin((2\pi/6)t) + 5 = 7.$$

Subtracting 5 from both sides, dividing by 3, and taking arcsin, we find

$$(2\pi/6)t = \arcsin(2/3)$$

which gives us

$$t = \frac{3}{\pi}\arcsin(2/3)$$

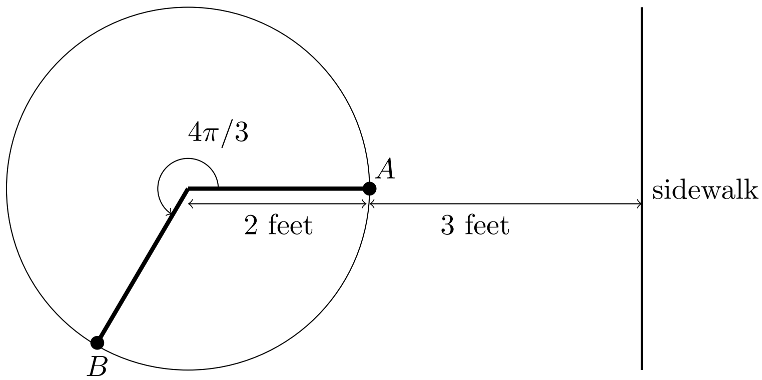
This is the first solution. We can find the second by using symmetries:

$$t = 3 - \frac{3}{\pi}\arcsin(2/3).$$

The third can be found by adding one period to the first solution:

$$t = 6 + \frac{3}{\pi}\arcsin(2/3).$$

6. [11 points] A duck in swimming in circles along the outer edge of a circular fountain in a park. The duck is 2 feet from the center of the fountain and swimming at a constant speed in a counter-clockwise direction. There is a sidewalk running north-south that passes 3 feet away from the fountain, as shown in the diagram below (which may not be drawn to scale). The duck starts at point A that is closest to the sidewalk. After 4 seconds, the duck is at the point B.



a. [2 points] How long does it take for the duck to make one full lap around the fountain? Include units.

Solution: $4\pi/3$ is $2/3$ of one full rotation, so a lap is $4/(2/3) = 4 \cdot 3/2 = 6$ seconds.

b. [3 points] How far did the duck travel along the circumference of the fountain between point A and point B? Give your answer in exact form and include units.

Solution: We know this is an arc with angle $\theta = 4\pi/3$ and radius is 2 feet, so the corresponding arc length is $r\theta = 2 \cdot 4\pi/3 = 8\pi/3$.

c. [6 points] Find a function $D(t)$ that gives the (horizontal) distance in feet between the duck and the sidewalk t seconds after the duck starts swimming.

Solution: From part (a), we know that this function will be periodic with period 6.
If we imagine a coordinate system with the origin at the center of the circle, then the x-coordinate of the duck's location is given by $2\cos(2\pi/6t)$, and so the distance from the sidewalk, at $x = 5$, to the duck, is found by taking the difference: $5 - 2\cos(2\pi/6t)$.
Another approach would be to note that since the duck is moving in a circle, this distance is a sinusoidal function moving between a minimum of 3 when $t = 0$ and a maximum of $3 + 2 + 2 = 7$. This means that the amplitude is 2 and the midline value is 5, and since the function starts at a minimum, we can use $-\cos(t)$ without a horizontal shift. This gives us $-2\cos(2\pi/6t) + 5$.