Math 105 — First Midterm — February 22, 2022

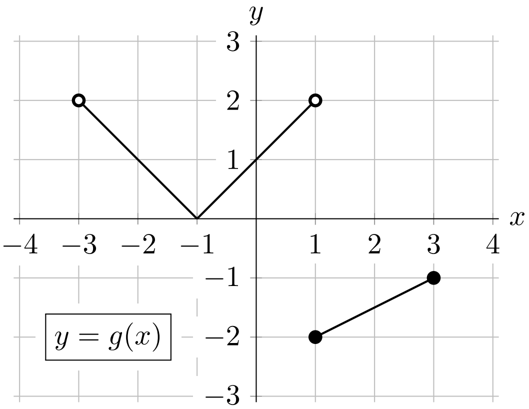
EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 7 pages including this cover. There are 6 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded. Scratchwork on pages other than those in this exam will not be graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. You must use the methods learned in this course to solve all problems.
8. You are allowed to use the notes written on two sides of a single $3'' \times 5''$ notecard. You may use any one calculator that does not have an internet or data connection.
9. However, you must show an appropriate amount of work for any calculation which we have learned how to do in this course, so that graders can see not only your answer but how you obtained it. If you use a graph or table to find an answer, be sure to sketch the graph or write out the table, and explain how the graph or table gives the answer.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is <u>not</u>.
11. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is <u>not</u> permitted.

Problem	Points	Score
1	7	
2	5	
3	8	

Problem	Points	Score
4	15	
5	8	
6	17	
Total	60	

1. [7 points] The entire graph of a function $g(x)$ is shown below to the left. Also shown is a table of some values for a different function $h(x)$. Assume that the function $h(x)$ is invertible.



x	-3	-1	0	1	3	4
$h(x)$	7	5	3	0	-2	-3

a. [3 points] Find the domain of $g(x)$ and range of $g(x)$. Give your answers using interval notation or using inequalities. *You do not need to explain or justify your answer.*

Answer: $g(x)$ has domain $(-3, 3]$ and range $[-2, -1] \cup [0, 2)$

b. [4 points] Find each of the following, or write N/A if a value does not exist or there is not enough information to find it. *You do not need to show work.*

i. $h^{-1}(-3)$

Answer: $h^{-1}(-3) =$ 4

ii. $g(h(0))$

Answer: $g(h(0)) =$ -1

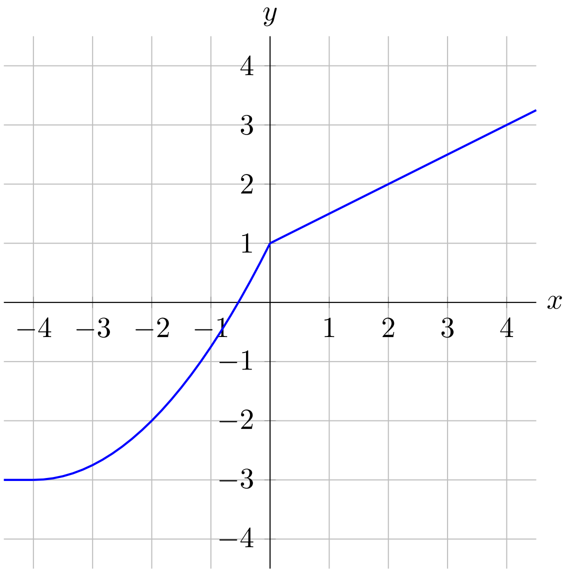
iii. all values of x so that $g(h(x)) = 1$

Answer: $x =$ 1, 3

2. [5 points] On the axes below, sketch the graph of a single possible function $y = f(x)$ satisfying all the listed properties.

- $f(0) = 1$
- the average rate of change of $f(x)$ on $[-4, 0]$ is 1
- $f(x)$ is concave up for $-4 < x < 0$
- $f(x)$ is invertible (that is, it has an inverse)
- $f(x)$ has a constant rate of change for $0 < x < 4$

Solution: One possible graph is shown.



3. [8 points] Jaime is on a long car trip. Consider the following functions:

- Let $d(t)$ be the distance, in miles, Jaime has driven t minutes after they begin their trip.
- Let $g(t)$ be the amount of gas, in gallons, in Jaime's car's gas tank t minutes after they begin their trip.

Assume that both functions have inverses. For each part below, write a phrase or sentence giving a practical interpretation of the given expression or equation, or explain why it doesn't make sense in this context.

a. $d(9) = 4$

Solution: When Jaime has driven for 9 minutes, they've gone 4 miles.

b. $g(d^{-1}(120))$

Solution: the amount of gas, in gallons, in Jaime's car's tank when they've driven 120 miles

c. $g(60) = g(0) - 2$

Solution: 60 minutes into their trip, Jaime's car has 2 fewer gallons of gas than when their trip started.

4. [15 points] Mei is starting a coffee roasting business.

a. [4 points] Mei puts green coffee beans into her roaster. Let $T(t)$ be the temperature, in degrees Fahrenheit (°F), inside the roaster t minutes after she starts roasting the beans. Some values of $T(t)$ are given in the table below.

t	0	3	5	12
$T(t)$	70	370	470	320

Compute the average rate of change of $T(t)$ over the interval $[0, 5]$. **Include units.**

Solution: $\dfrac{470-70}{5-0} = \dfrac{400}{5} = 80$

Answer: **80°F per minute**

Could $T(t)$ be concave down on the entire interval $[0, 12]$? Show your work, and circle your final answer.

Solution: The average rates of change over the three consecutive subintervals are $\dfrac{370-70}{3-0} = 100$, $\dfrac{470-370}{5-3} = 50$, and $\dfrac{320-470}{12-5} < 0$. Since these are decreasing, yes, the function could be concave down on this interval.

Answer (*circle one*): [✓ **Yes**] **No**

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Mei is starting a coffee roasting business.

b. [3 points] Let n be a variable representing the number of customers that come into her shop on the dth day it is open (so that $d = 1$ represents the first day she is open, etc.). Is it definitely true that d is a function of n? Briefly explain your answer.

Answer (*circle one*):

Yes, d must be a function of n [✓ **No**, d might not be a function of n]

Explanation:

Solution: It could be that on different days she is open, the same number of customers come into her shop, which would mean one input value (number of customers n) would have more than one output (day d she is open).

c. [5 points] Mei plans to sell her roasted coffee beans for \$15 per pound. However, she plans to offer a deal: once a customer has spent \$60 on coffee beans, any additional beans will only cost \$12 per pound. Find a piecewise-defined formula for $C(p)$, the cost to purchase p pounds of Mei's coffee beans.

Answer: $$C(p) = \begin{cases} 15p & \text{for } 0 \le p \le 4 \\ 60 + 12(p-4) & \text{for } p > 4 \end{cases}$$

d. [3 points] Compute $C^{-1}(75)$. Then, using a complete sentence and **including units**, give a practical interpretation of your answer in the context of the problem.

Solution: Because we need to find the value of p so that $C(p) = 75$, we know this will be for some $p > 4$, so we set the second piece of our function equal to 75. Solving, we find that

$$\begin{aligned} 75 &= 60 + 12(p-4) \\ \frac{15}{12} &= (p-4) \\ p &= \frac{5}{4} + 4 = \frac{21}{4} = 5.25 \end{aligned}$$

Answer: $C^{-1}(75) =$ **5.25**

Interpretation:

Solution: If a customer spent \$75 on coffee beans, they purchased 5.25 pounds of beans.

5. [8 points] For each part of this problem, you must **show every step** of any algebraic work that is required.

a. [3 points] The quadratic function $q(x)$ has its vertex at the point $(2, 4)$ and a zero at $x = 5$. Find a formula for $q(x)$.

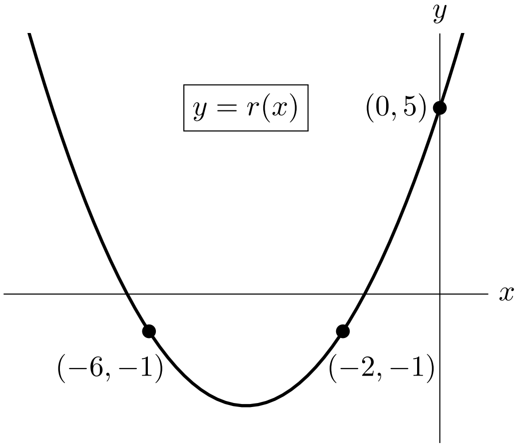
Solution: There are at least two possible solutions:

Using vertex form: we know that $q(x) = a(x-2)^2 + 4$ for some a. Then plugging in the point $(5, 0)$, we find $0 = a(5-2)^2 + 4 = 9a + 4$, so $a = \frac{-4}{9}$.

Using factored form: we know that, by symmetry, the function must have another zero at $x = -1$. Then we know that $q(x) = a(x-5)(x+1)$ for some a. Then plugging in the point $(2, 4)$, we find $4 = a(2-5)(2+1) = -9a$, so $a = \frac{-4}{9}$.

Answer: $q(x) =$ $\frac{4}{9}(x-2)^2 + 4$ **or** $\frac{4}{9}(x-5)(x+1)$

b. [5 points] Find a formula for the quadratic function $r(x)$ graphed below.



Solution:
There are at least three possible solutions:

If we first consider the function $p(x) = r(x) + 1$ by shifting the given graph up by 1, we can write

$$p(x) = a(x+6)(x+2).$$

Then noting that $(0, 6)$ is a point on $p(x)$, we have

$$6 = a(6)(2),$$

so that $a = \frac{1}{2}$. Then we know

$$r(x) = p(x) - 1 = \frac{1}{2}(x+6)(x+2) - 1.$$

We could also find a formula for $r(x)$ by writing $r(x) = a(x+4)^2 + k$, since we know by symmetry that the x-coordinate of the vertex is 4. Then we can plug in $(0, 5)$ and one of the other known points to solve a system of two equations for $a = \frac{1}{2}$ and $k = -3$. The formula obtained from this method is $r(x) = \frac{1}{2}(x+4)^2 - 3$.

Or, because we know the y-intercept is 5, we can write $r(x) = ax^2 + bx + 5$, and proceed similarly to the solution above using the other two points to solve for $a = \frac{1}{2}$ and $b = 4$. The formula obtained from this method is $r(x) = \frac{1}{2}x^2 + 4x + 5$.

Answer: $r(x) =$ $\frac{1}{2}(x+6)(x+2) - 1$

6. [17 points] A scientist is studying the mass, in milligrams (mg), of several different bacterial colonies.

- Colony A's mass is 17 mg at the start of the experiment, and it grows at a rate of 7% per hour.
- Colony B's mass in mg t hours after the experiment begins is given by $B(t) = 3e^{0.11t}$.
- Colony C's mass in mg t hours after the experiment begins is given by $C(t) = 22(1.04)^t$.
- Two hours into the experiment, Colony D has a mass of 21 mg, but by four hours into the experiment, only 18 mg remains.

For each part of this problem, you must **show every step** of any algebraic work that is required.

a. [3 points] Find a formula for the function $A(t)$, which gives the mass, in mg, of colony A t hours after the experiment begins.

Answer: $A(t) =$ $17(1.07)^t$

b. [2 points] By what percent is colony B growing each hour? Give your answer in exact form or rounded to at least two decimal places.

Answer: $100(e^{0.11} - 1) \approx 11.63$ %

c. [3 points] How many hours will it take for colony B's population to triple?
Give your answer in exact form, and circle your final answer.

Solution:

$$\begin{aligned} 3e^{0.11t} &= 9 \\ e^{0.11t} &= 3 \\ 0.11t &= \ln(3) \\ t &= \frac{\ln(3)}{0.11} \end{aligned}$$

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- Two hours into the experiment, Colony D has a mass of 21 mg, but by four hours into the experiment, only 18 mg remains.

For each part of this problem, you must **show every step** of any algebraic work that is required.

d. [5 points] At what time t will the size of colonies B and C be the same? *Give your answer in exact form, and circle your final answer.*

Solution: There are at least two solutions:

We can take the natural log of both sides first:

$$\begin{aligned} 3e^{0.11t} &= 22(1.04)^t \\ \ln(3) + \ln(e^{0.11t}) &= \ln(22) + \ln(1.04)^t \\ \ln(3) + 0.11t &= \ln(22) + t\ln(1.04) \\ 0.11t - t\ln(1.04) &= \ln(22) - \ln(3) \\ t(0.11 - \ln(1.04)) &= \ln(22) - \ln(3) \\ t &= \frac{\ln(22) - \ln(3)}{0.11 - \ln(1.04)} \end{aligned}$$

We can also rearrange the equation first:

$$\begin{aligned} 3e^{0.11t} &= 22(1.04)^t \\ \frac{e^{0.11t}}{(1.04)^t} &= \frac{22}{3} \\ \left(\frac{e^{0.11}}{1.04}\right)^t &= \frac{22}{3} \\ t\ln\left(\frac{e^{0.11}}{1.04}\right) &= \ln\left(\frac{22}{3}\right) \\ t &= \frac{\ln\left(\frac{22}{3}\right)}{\ln\left(\frac{e^{0.11}}{1.04}\right)} \end{aligned}$$

e. [4 points] Assuming colony D's mass is decaying exponentially, what will its mass (in mg) be 12 hours after the start of the experiment? *Give your answer in exact form.*

Solution: We can plug in the points $(2, 21)$ and $(4, 18)$ into $y = ab^t$, giving $21 = ab^2$ and $18 = ab^4$. Dividing, this leads to $\frac{18}{21} = b^2$, or $b = \left(\frac{6}{7}\right)^{1/2}$.

Then we can solve for a by plugging b into, say, $21 = ab^2$, which leads to $a = \frac{21}{\frac{6}{7}} = \frac{21 \cdot 7}{6} = 24.5$. Finally, we can find the mass when $t = 12$ by using the values we found for a and b along with $t = 12$:

$$24.5\left(\frac{6}{7}\right)^6$$

Alternately, once we have b, we can instead note that the mass after 12 hours will be 18, the mass at 4 hours, times b^8.

Answer: $24.5\left(\frac{6}{7}\right)^6$