Math 105 — Second Midterm — April 12, 2022

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 0 pages including this cover. There are 0 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded. Scratchwork on pages other than those in this exam will not be graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. You must use the methods learned in this course to solve all problems.
8. You are allowed to use the notes written on two sides of a single $3'' \times 5''$ notecard. You may use any one calculator that does not have an internet or data connection.
9. However, you must show an appropriate amount of work for any calculation which we have learned how to do in this course, so that graders can see not only your answer but how you obtained it. If you use a graph or table to find an answer, be sure to sketch the graph or write out the table, and explain how the graph or table gives the answer.
10. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
11. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

Problem	Points	Score
1		
2		
3		

Problem	Points	Score
Total		

1. [0 points]

a. [4 points] Let $f(x)$ be an **odd**, periodic function with period 6. Some values for $f(x)$ are given below.

x	-2	-1	0	1	2
$f(x)$	-5	a	b	-3	5

Find the following, or write NEI if there is not enough information provided to do so:

i. $a =$ **3**

ii. $b =$ **0**

iii. $f(4) =$ **5**

iv. $f(f(2)) =$ -3

b. [4 points] Suppose that $h(x)$ is an **even**, periodic function with period 4, amplitude 7, and midline $y = -2$. Define

$$j(x) = -3h\left(\frac{1}{2}x\right).$$

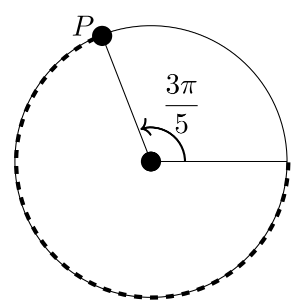
Is $j(x)$ even, odd, or neither? Circle the one correct answer.

[EVEN] ODD NEITHER

Find the period, amplitude, and midline of $j(x)$:

Period: 8 **Amplitude:** 21 **Midline:** $y = 6$

2. [0 points] Consider the diagram shown to the right.



a. [2 points] Find the exact value of another angle θ, in radians, with $0 \leq \theta \leq 2\pi$, such that the value of $\cos(\theta)$ is the same as the value of $\cos\left(\frac{3\pi}{5}\right)$.

Answer: $\theta =$ $7\pi/5$

Now suppose that the circle shown is centered at the point $(-2, 1)$ and has radius 7.

b. [4 points] Find the x- and y-coordinates of the point P.

Answer: $(x, y) =$ $(3\cos(7\pi/5) - 2, 7\sin(3\pi/5) + 1)$

c. [3 points] Find the arclength of the bold, dashed arc going from the point P counterclockwise to the right-most point of the circle.

Answer: $14\pi - 7 \cdot 3\pi/5 \;=\; 7 \cdot 7\pi/5$

3. [0 points] The strawberries at Maggie's farm are ready to be picked. Her friend Arun is willing to help out.

- Let $M(t)$ be the amount of strawberries, in pounds, that Maggie can pick in t minutes.
- Let $A(t)$ be the amount of strawberries, in pounds, that Arun can pick in t minutes.

Assume that both of these functions have inverses.

a. [5 points] For parts i. and ii. below, write a complete sentence giving a practical interpretation of the given equation.

i. $M^{-1}(2) = 10$

Solution: Maggie can pick 2 pounds of strawberries in 10 minutes.

ii. $M(A^{-1}(5)) = 8$

Solution: In the time Arun can pick 5 pounds of strawberries, Maggie can pick 8 pounds of strawberries.

b. [3 points] Suppose that, together, Maggie and Arun pick P pounds of strawberries in total. If Arun picked strawberries for 180 minutes, write an expression for the time, in minutes, that Maggie picked strawberries. *Your answer may involve the quantity P, but you should* ***not*** *assume that Maggie and Arun picked strawberries for equal amounts of time.*

Answer: $M^{-1}(P - A(180))$

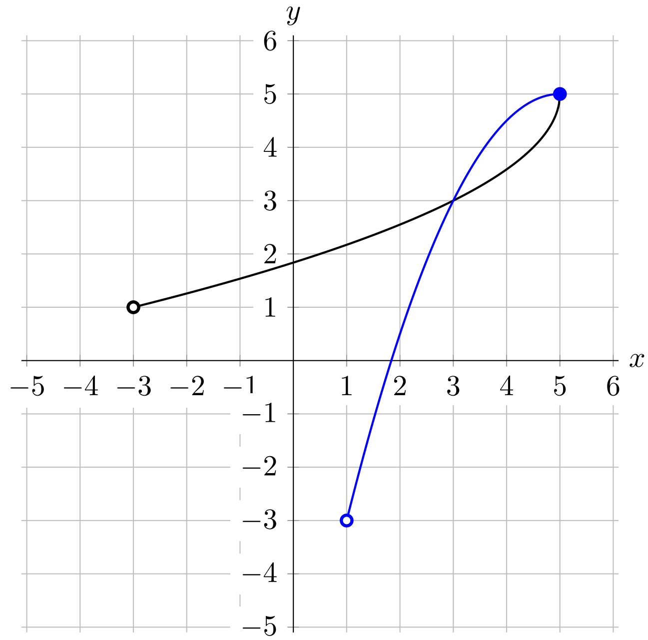
c. [3 points] Define the function $N(s)$ to be the amount of strawberries, in **ounces**, that Maggie can pick in s **hours**. Write a formula for $N(s)$ in terms of M. *There are 16 ounces in a pound.*

Answer: $N(s) =$ $16M(60s)$

4. [5 points] Given below is the graph of a function $B(x)$. Briefly explain how you can tell that the function $B(x)$ is invertible. Then, on the same set of axes, carefully sketch the graph of $B^{-1}(x)$.

Explanation:

Solution: The function $B(x)$ is invertible because it passes the horizontal line test; or, because each output of $B(x)$ corresponds with only one input x.



5. [0 points] Suppose that $a(x) = mx + b$ for some constants m and b, where both m and b are not equal to zero.

In parts **a.** and **b.**, decide whether each of the following statements must be true, or whether it could be false, and circle the appropriate answer. You do not need to show work but limited partial credit may be available for work shown.

a. [2 points] If $f(x) = 3x - 5$, then the function $f(x) + a(x)$ must be linear.

[TRUE] FALSE

Solution: Note that $f(x)+a(x) = 3x-5+mx+b = (3+m)x+(b-5)$, which is also linear since m and b are constants.

b. [2 points] If $f(x) = 3x - 5$, then the function $f(x) \cdot a(x)$ must be linear.

TRUE [FALSE]

Solution: Now $f(x) \cdot a(x) = (3x-5)(mx+b)$, which is quadratic since it will have a $3mx^2$ term if expanded, and $m \neq 0$.

Also define the function $q(x) = x^2 + 3$.

c. [3 points] If $a(q(x)) = \frac{1}{3}x^2$, find the values of m and b. **Show all work.**

Do not use these values of m and b for the other parts of this problem.

Solution:

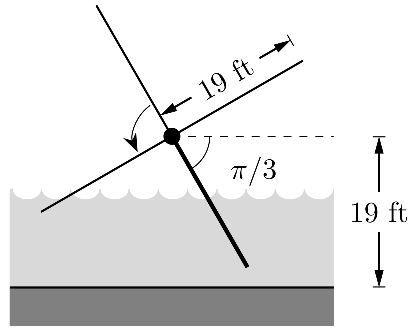
Using the given formulas for a and q, we first find the composition $a(q(x)) = a(x^2+3) = m(x^2+3)+b = mx^2+3m+b$. Since we're told this must equal $\frac{1}{3}x^2$, we must have that $m = 1/3$. Then, since $3m+b$ must be 0, we can solve $3(1/3)+b=0$, or $b=-1$.

$m =$ 1/3

$b =$ −1

6. [0 points] You are standing by a river, watching two water wheels, each of which is rotating counterclockwise at a different but constant speed.

The first water wheel takes 24 seconds to complete a full revolution. Each blade of the wheel is 19 feet long, and when each blade is at its lowest point, it just barely scrapes the bottom of the river. One of the blades is painted red, shown as the thicker blade in the diagram to the right.



At the moment you begin watching, the red blade is exactly $\frac{\pi}{3}$ radians below the horizontal, as depicted. Let $r(t)$ be the height, in feet, of the tip of the red blade above the bottom of the river t seconds after you begin watching.

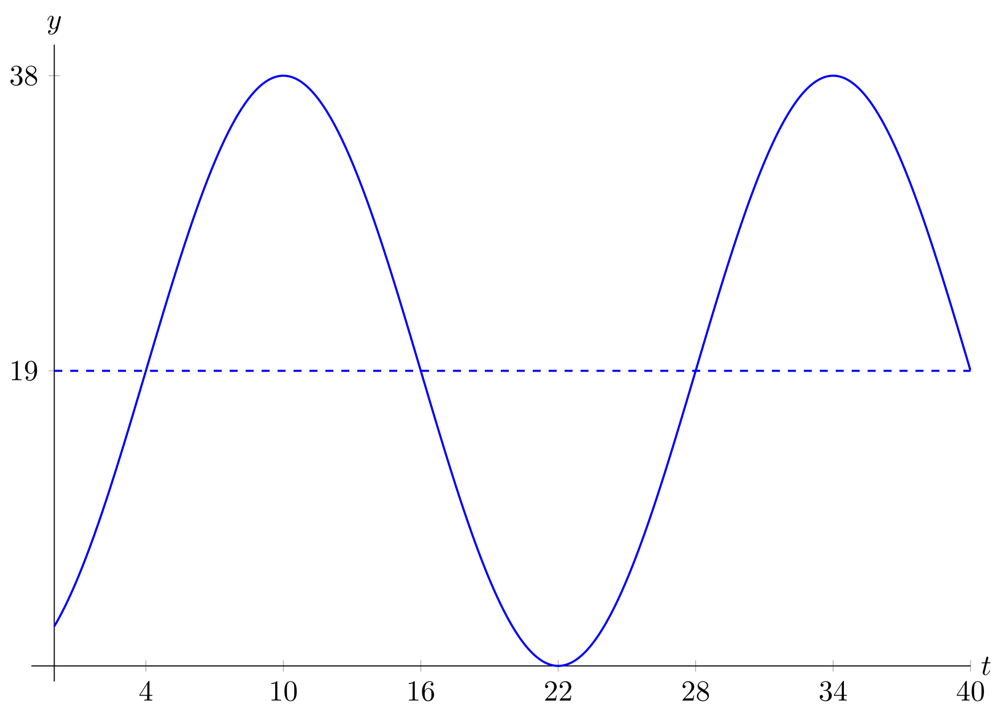
a. [1 point] How many seconds does it take the red blade to reach the horizontal position?

Solution: Since the blade starts $\pi/3$ radians below the horizontal and is moving counterclockwise, it needs to move $1/6$ of a full rotation to reach the horizontal position. Since a full rotation is 24 sections, this will take $24/6 = 4$ seconds.

Answer: 4

b. [4 points] Sketch a graph of $y = r(t)$ on the interval $0 \leq t \leq 40$. Be sure the scales on your axes are clear, and pay careful attention to the shape of your graph.

Solution: Given part a, the graph should start below the midline of $y = 19$ and reach the midline at $t = 4$. Note that he period of the graph is 24, and the amplitude is 19.



c. [4 points] Find a formula for $r(t)$.

Answer: $r(t) = 19\sin(2\pi/24(t-4)) + 19$ **or** $19\sin(2\pi t/24 - \pi/3) + 19$

This problem continues onto the following page.

This problem continues from the previous page and is restated for your convenience.

You are standing by a river, watching two water wheels, each of which is rotating counterclockwise at a different but constant speed.

d. [5 points] The second water wheel has a blade painted blue, and you have determined that the height, in feet, of the tip of this blade above the bottom of the river t seconds after you began watching is given by

$$20 + 15\sin\left(\frac{\pi}{8}t\right).$$

Find the **first three** positive values t for which the height of the blade is 30 feet. **Show your work**, and give your answers in **exact** form.

Solution: We need to find when the height is 30, so we set the given formula equal to 30 to find one solution:

$$20 + 15\sin\left(\frac{\pi}{8}t\right) = 30$$

$$15\sin\left(\frac{\pi}{8}t\right) = 10$$

$$\sin\left(\frac{\pi}{8}t\right) = \frac{2}{3}$$

$$\frac{\pi}{8}t = \arcsin\left(\frac{2}{3}\right)$$

$$t = \frac{8}{\pi}\arcsin\left(\frac{2}{3}\right).$$

This will give us the first positive solution since $2/3 > 0$. Then, by using a sketch of $\sin\left(\frac{\pi}{8}t\right)$ (or of the given formula for the height), which has a period of 16, we can see that the second positive solution can be found by taking 8 minus the first solution. Finally, the third positive solution will be one period to the right of the first solution.

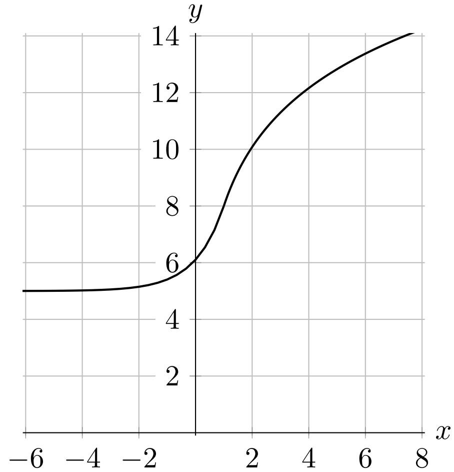
Answers: $\frac{8}{\pi}\arcsin\left(\frac{2}{3}\right)$

$8 - \frac{8}{\pi}\arcsin\left(\frac{2}{3}\right)$

$16 + \frac{8}{\pi}\arcsin\left(\frac{2}{3}\right)$

7. [6 points] Consider the piecewise-defined function $k(x)$ given below. A portion of the graph of $k(x)$ is also shown for reference.

$$k(x) = \begin{cases} 3e^{x-1} + 5 & x < 1 \\ 3\ln(x) + 8 & x \geq 1 \end{cases}.$$



Find a formula for $x = k^{-1}(y)$. Be sure to **show your work**.

Solution: We set the first piece equal to y and solve for x:

$$3e^{x-1} + 5 = y$$
$$e^{x-1} = \frac{y-5}{3}$$
$$x - 1 = \ln\left(\frac{y-5}{3}\right)$$
$$x = \ln\left(\frac{y-5}{3}\right) + 1.$$

To find where this piece of the inverse function will apply, we look at the output values for $k(x)$ for $x < 1$, which is $5 < y < 8$.
Now we set the second piece equal to y and solve for x:

$$3\ln(x) + 8 = y$$
$$\ln(x) = \frac{y-8}{3}$$
$$x = e^{(y-8)/3}.$$

To find where this piece of the inverse function will apply, we look at the output values for $k(x)$ for $x \geq 1$, which is $y \geq 8$.

Answer: $k^{-1}(y) = \begin{cases} \ln\left(\frac{y-5}{3}\right) + 1 & \text{for } 5 < y < 8 \\ e^{(y-8)/3} & \text{for } 8 \leq y \end{cases}$