Math 105 — First Midterm — February 13, 2023

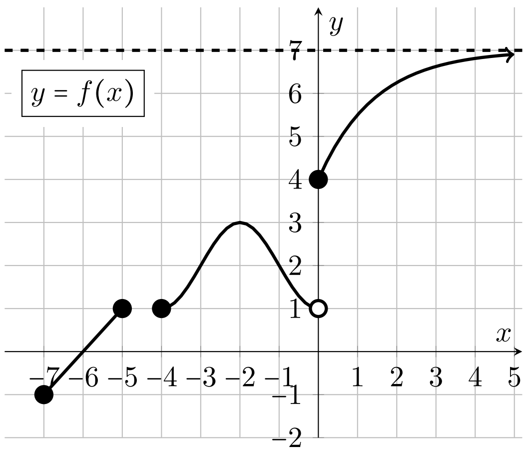
EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. Use a pencil for "bubble-in" questions so that you can easily erase your answer if you change your mind.
4. This exam has 12 pages including this cover. There are 7 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
5. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
6. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
7. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
8. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
9. You must use the methods learned in this course to solve all problems.
10. You are allowed notes written on two sides of a $3'' \times 5''$ note card and one scientific calculator that does not have graphing or internet capabilities.
11. If you use a graph or table to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how the graph or table gives the answer.
12. Include units in your answer where that is appropriate.
13. Problems may ask for answers in *exact form* or in *decimal form*. Recall that $\sqrt{2} + \cos(3)$ is in exact form and $x = 0.424$ would be the same answer expressed in decimal form.
14. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

Problem	Points	Score
1	9	
2	8	
3	11	
4	11	

Problem	Points	Score
5	7	
6	6	
7	8	
Total	60	

1. [9 points] Part of the graph of a function $f(x)$ is shown below to the left; note that it has a **horizontal asymptote of** $y = 7$. Also shown is a table of some values for an invertible function $g(x)$, and formula for a function $h(x)$.



x	-5	-2	-1	0	1	2
$g(x)$	6	-5	0	4	7	9

$$h(x) = \begin{cases} x^2 + 1, & 0 \le x < \infty \\ x + 1, & -\infty < x < 0 \end{cases}$$

a. [3 points] Find the domain and range of $f(x)$. Give your answers using interval notation or using inequalities. *You do not need to explain or justify your answer.*

Domain: $[-7, -5] \cup [-4, \infty]$

Range: $[-1, 3] \cup [4, 7)$

b. [6 points] Find or estimate the value of each of the following; write N/A if a value does not exist or there is not enough information to find it.
You do not need to show work.

(i) $g(f(-1)) =$ $g(2) = 9$

(ii) $f(g^{-1}(-5)) =$ $f(-2) = 3$

(iii) $h^{-1}(-5) =$ -6

(iv) $g(h(-2)) =$ $g(-2+1) = g(-1) = 0$

(v) $\lim_{x \to \infty} f(x) =$ $\mathbf{7}$

(vi) If $q(x) = g(x - 3) + 2$, $q(2) =$ $g(2-3)+2 = g(-1)+2 = 0+2 = 2$

2. [8 points] The UM Etsy Club is 3D printing a new bracelet design called the Helix Monster. The cost of the materials for one bracelet depends on the inner circumference of that bracelet. The cost of materials B (in dollars) for a Helix Monster bracelet with an inner circumference of c centimeters is given by:

$$B = h(c) = 2 + 0.4c$$

a. [2 points] If the club members want to spend at most $12 in materials on a Helix Monster bracelet, what is the largest the bracelet's inner circumference could be? *Include units.*

Solution:

$$\begin{aligned} 12 &= 2 + 0.4c \\ 10 &= 0.4c \\ 10/0.4 &= c \\ 25 &= c \end{aligned}$$

$c = \frac{12-2}{0.4} = 25$ **cm**

b. [3 points] Another member creates a Swirling Storm design that has different production costs. The cost (B, in dollars) to produce one Swirling Storm design with inner circumference c is given by

$$B = s(c) = 2.5 + 0.25c$$

For what values of c does the Helix Monster design cost less? For what values of c does the Swirling Storm design cost less? *Express your answers using inequalities or interval notation below. Show all work. No explanation needed.*

Solution: To find when one bracelet switches from being cheaper to more expensive, we need to find at which value of c the two cost functions intersect.

$$\begin{aligned} 2 + 0.4c &= 2.5 + 0.25c \\ 0.15c &= 0.5 \\ c &= 0.5/0.15 = 10/3 \end{aligned}$$

Because the Helix Monster starts out cheaper ($2 vs. $2.50), we know it will be cheaper for smaller values of c. After they intersect, Swirling Storm will be cheaper.

Helix Monster is cheaper when: $0 \le c < 10/3$ **cm**

Swirling Storm is cheaper when: $c > 10/3$ **cm**

c. [3 points] The club decides to produce a large batch of Swirling Storm bracelets with inner circumference 24cm. The price to rent the printer for the day is $120. Write an expression for the total cost T (in dollars) for producing n Swirling Storm bracelets for inner circumference 24cm.

Solution: The cost for each Swirling Storm bracelet with an inner circumference of 24cm is

$$s(24) = 2.5 + 0.25 \cdot 24 = 8.5.$$

So the cost to produce n such bracelets, including the cost of the printer rental, will be :

$$120 + 8.5n$$

$T =$ $120 + 8.5n$

3. [11 points] The UM Dance Club met up with the UM Math Modeling Club to write formulas for different dancer's jumps. They measure one dancer's total time in the air as 1 second and their maximum height as 4 feet. They know that the function $D(t)$ which gives the dancer's height (in feet) as a function of time after they jump (in seconds) is a quadratic function.

a. [3 points] One member of the Math Modeling Club wants to find the formula for $D(t)$ using the zeros of the function, so is starting with the form:

$$D(t) = a(t - r)(t - s)$$

To model the dancer's jump described above, what are possible values of r and s and how do you know?

$r =$ **0**

$s =$ **1**

Explanation:

Solution: The zeros of this function are when the dancer's height is 0. If the dancer's total time in the air is 1 second, that means that they are on the ground (height 0) at $t = 0$ seconds and then again 1 second later at $t = 1$ second. Since these are the zeros of $D(t)$, they are the values of r and s in the factored form taken as the starting point.

b. [3 points] Another member of the Math Modeling Club wants to write a formula using vertex form of a quadratic function:

$$D(t) = a(t - h)^2 + k$$

To model the dancer's jump described above, what are the values of h and k in this formula and how do you know?

$h =$ **0.5**

$k =$ **4**

Explanation:

Solution: $D(t)$ is given in vertex form this time, so (h, k) is the coordinates of the vertex. Because the zeros are at $t = 0, 1$, the t-value of the vertex must be halway between them, at $t = 0.5$. This is why $h = 0.5$. We're also told that the dancer's maximum height is 4ft, so this gives us the vertical coordinate of the vertex, and $k = 4$.

The UM Dance Club met up with the UM Math Modeling Club to write formulas for different dancer's jumps. They measure one dancer's total time in the air as 1 second and their maximum height as 4 feet. They know that the function $D(t)$ which gives the dancer's height (in feet) as a function of time after they jump (in seconds) is a quadratic function.

c. [3 points] Find the exact value of a in the formulas above. *You can use either of your equations to do this. Show all work.*

Solution: We can solve this using either starting point: factored form OR vertex form. For the sake of the solutions, we show both ways. (This is also a way to verify that our work is correct!). For both methods, we'll plug in an additional known point into our starting equations, and then solve for the value of a.

Method 1: plug in vertex $(0.5, 4)$ into factored form from part (a) and solve for a.

$$\begin{aligned} 4 &= a(0.5-0)(0.5-1) = a(0.5)(-0.5) \\ 4 &= a(-0.25) \\ 4/-0.25 &= a \\ -16 &= a \end{aligned}$$

Method 2: plug in either zero ($(0,0)$ or $(1,0)$) into vertex form from part (b) and solve for a.

$$\begin{aligned} 0 &= a(0-0.5)^2 + 4 \\ -4 &= a(-.5)^2 \\ -4 &= a(0.25) \\ -4/0.25 &= a \\ -16 &= a \end{aligned}$$

$a =$ **-16**

d. [2 points] From the context of the problem alone—without relying on or referring to your calculation above—would you expect the value of a to be positive or negative? Why?

$a > 0$ $\boxed{a < 0}$ NOT ENOUGH INFORMATION

Explanation:

Solution: Because this dancer reaches a maximum height, this must be a "downward facing" or concave-down parabola. For such a parabola, the leading coefficient, a, must be negative.

4. [11 points] The UM Youtubers Club makes a very cool new video that goes viral. Suppose the video had 100 views at 2:00AM Eastern Time (ET) Saturday morning and the number of views grew exponentially for at least the next 24 hours, with views doubling every hour.

a. [1 point] Between 3:00AM ET and 6:00AM ET Saturday, by what factor had the number of views increased?

Solution: Because the number of views doubles each hours, over three hours it will double three times. In other words, it will increase by a factor of $2^3 = 8$.

Factor of increase: $2^3 = 8$

b. [2 points] Write a formula for a function $V = f(t)$, where V is the number of video views and t is the number of hours since 2:00AM ET Saturday.

Solution: We know the number of views at 2:00AM is 100, and that it doubles each hour after that. That means the growth factor is $b = 2$. Putting this together we get the exponential function $V = f(t) = 100 \cdot 2^t$.

$V = f(t) =$ $100 \cdot 2^t$

c. [6 points] For each of the following expressions or equations, explain its meaning in the context of the problem, or explain why it doesn't make sense in the context of the problem.

(i) $f^{-1}(500,000) \approx 12.25$

Solution: The time at which the number of views reaches 500,000 is approximately 12.25 hours after 2AM, or 2:15PM.

(ii) $\frac{f(5)-f(3)}{5-3} = 1200$

Solution: Between 5AM and 7AM, the views increased, on average, by 1200 views per hour.

d. [2 points] Write a new function $g(s)$ **in terms of** f that would give us the number of views the video had s hours after 9:00AM ET on Saturday morning.

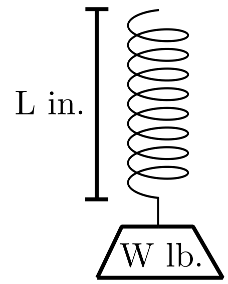
Solution: Graphical perspective: if our new starting point is 9:00AM, this is like shifting our graph so what was previously at $t = 7$ is now at $s = 0$. So this is, graphically speaking, a shift left by 7. This means our new formula is $g(s) = f(s + 7)$.

Points perspective: If I put $s = 0$ (9AM) into our new function g, this should give the same output as $t = 7$ (9AM) into our original function $f(t)$. This examples shows that we need to add 7 to our s values before putting them into f. This yields the same result as above: $g(s) = f(s + 7)$.

$g(s) =$ $f(s + 7)$

5. [7 points] The UM Weights and Measures Club is building a spring scale, which weighs objects by hanging them from a spring.

Let W be the weight of an object, in pounds, and let L be the length of the spring in inches when we hang that object from it. It turns out that there is a linear relationship between W and L. The club observes that their spring is 3 inches long when no weight is attached, and stretches out to 5.5 inches when they test it with a 5-pound weight.



a. [3 points] What is the slope of the function $W = f(L)$? Explain the meaning of the slope's value in the context of the problem.

Solution: The slope is "change in input divided by change in output." In this case, we're told that the weight (output) changes by 5 lbs and the length of the spring (input) changes by $5.5 - 3 = 2.5$ inches. This means our slope is $5/2.5 = 2$ pounds per inch.

Another perspective: We know two points on the graph of this function: $(3, 0)$ and $(5.5, 5)$. Using those two poitns we can find the slope of the line between them and arrive at the same answer as above.

Slope = **2**

Meaning:

Solution: Our slope's units of "pounds per inch" (coming from "change in output / change in input") are useful here! "2 pounds per inch" mean that for each additional inch the spring stretches, there had been 2 more pounds added to the scale.

b. [2 points] Find a formula for $W = f(L)$.

Solution: Because we know the slope is 2 and we know that $(3, 0)$ is on the graph of the function, we can use point-slope form to find our equation:

$$W = 2(L - 3) + 0 = 2(L - 3) = 2L - 6$$

We get the same formula if we use the other known point $(5.5, 5)$:

$$W = 2(L - 5.5) + 5 = 2L - 11 + 5 = 2L - 6$$

W = $2L - 6$

c. [2 points] Suppose we hang a bucket from the spring and then pour in some water. As we add the weight of the water, the spring gets 4 inches longer. How much does the added water weigh? *Include units.*

Solution: Since we know the slope is 2 lbs / inch, this means that for each inch longer the spring grows, there was a corresponding addition of 2 lbs of weight. So if the spring got 4 inches longer, that came from added weight of $4 \times 2 = 8$ pounds.

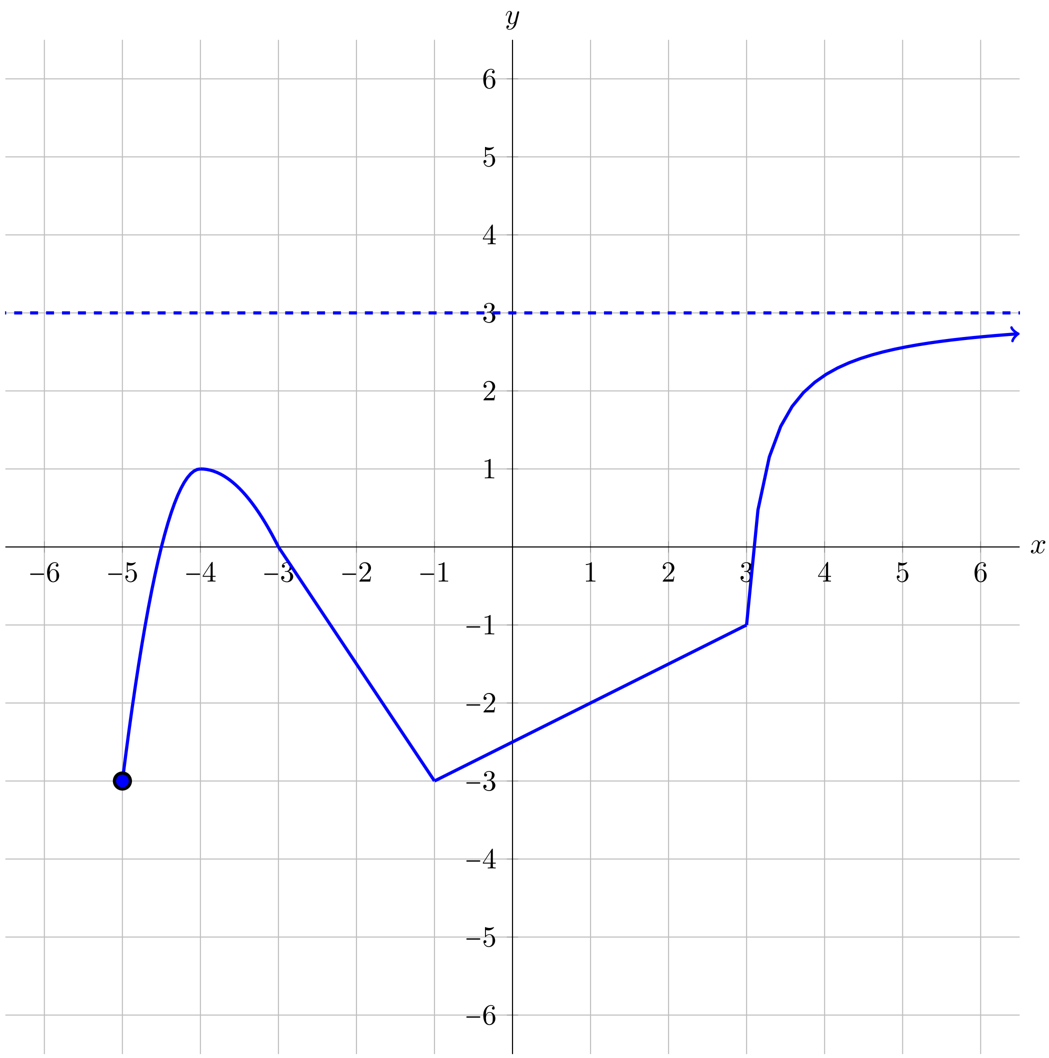
Note that is doesn't work to plug 4 inches into our formula for W because the problem is not saying the spring was 4 inches total, but saying that it *lengthened* by 4 inches. That is, the change is four inches, but not the total.

The water in the bucket weighs ____**8 pounds**____

6. [6 points] On the axes below, sketch the graph of a single function $y = f(x)$ with all of the following properties:

- The domain of $f(x)$ is $-5 \leq x < \infty$.
- The range of $f(x)$ is $-3 \leq x < 3$.
- $f(x)$ is concave down on the interval $-5 \leq x \leq -3$.
- $f(x)$ is decreasing on the interval $-4 \leq x \leq -1$.
- The average rate of change of $f(x)$ between $x = -1$ and $x = 3$ is $1/2$.
- $f(x) \to 3$ as $x \to \infty$

Note: there are many possible solutions.



7. [8 points] The following table shows some values of 3 different functions:

x	6	7	9
$f(x)$	27	18	8
$g(x)$	15	20	25
$h(x)$	12	16	24

This blank space is for your work and calculations.

Solution:

Examine function f:

- Average rate of change of f over $[6,7]$: -9; average rate of change of f over $[7,9]$: $-10/2 = -5$. These are not the same, so f cannot be linear.
- f is decreasing
- These average rates of change are increasing as we go left to right, so f could be concave up.
- From $x = 6$ to $x = 7$, $f(x)$ changes by a factor of $18/27 = 2/3$. To see if that same growth factor is in play from $x = 7$ to $x = 9$, we need to apply it *twice*.

$$\frac{2}{3} \cdot \frac{2}{3} \cdot f(7) = \frac{2}{3} \cdot \frac{2}{3} \cdot 18 = \frac{2}{3} \cdot 12 = 8 = f(9)$$

This means the same growth factor holds from $x = 7$ to $x = 9$, so f could be exponential with growth factor of $2/3$.

Examine function g:

- Average rate of change of g over $[6,7]$: 5; average rate of change of g over $[7,9]$: $5/2 = 5/2$. These are not the same, so g cannot be linear.
- g is increasing
- These average rates of change are decreasing as we go left to right, so f could be concave down (but not concave up).
- From $x = 6$ to $x = 7$, $g(x)$ changes by a factor of $20/15 = 4/3$. To see if that same growth factor is in play from $x = 7$ to $x = 9$, we need to apply it *twice*.

$$\frac{4}{3} \cdot \frac{4}{3} \cdot g(7) = \frac{4}{3} \cdot \frac{4}{3} \cdot 20 = \frac{4}{3} \cdot \frac{80}{3} = \frac{320}{3} \neq g(9) = 25$$

This means that the $\frac{4}{3}$ growth factor does not continue to hold, so $g(x)$ cannot be exponential.

Examine function h:

- Average rate of change of h over $[6,7]$: 4; average rate of change of h over $[7,9]$: $8/2 = 4$. These are the same, so h could be linear.
- h has a positive slope and is increasing
- Since the average rates of change are *constant*, they are neither increasing or decreasing, so $h(x)$ can be neither concave down nor concave up.
- Since $h(x)$ has a constant rate of change, it cannot be exponential (since all exponential functions have changing rates of change).

Circle **all** correct options for each part.

a. [2 points] Which of these functions could be linear?

$f(x)$ $g(x)$ $\boxed{h(x)}$ NONE OF THESE

b. [2 points] Which of these functions could be exponential?

$\boxed{f(x)}$ $g(x)$ $h(x)$ NONE OF THESE

c. [2 points] Which of these functions could be concave up on the interval $[6, 9]$?

$\boxed{f(x)}$ $g(x)$ $h(x)$ NONE OF THESE

d. [2 points] Which of these functions could be increasing on the interval $[6, 9]$?

$f(x)$ $\boxed{g(x)}$ $\boxed{h(x)}$ NONE OF THESE