Math 105 — Second Midterm — March 20, 2023

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 14 pages including this cover. There are 7 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are allowed notes written on two sides of a $3'' \times 5''$ note card and one scientific calculator that does not have graphing or internet capabilities.
10. If you use a graph or table to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how the graph or table gives the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

Problem	Points	Score
1	10	
2	10	
3	11	
4	11	

Problem	Points	Score
5	6	
6	7	
7	5	
Total	60	

1. [10 points] Below is a table giving some values of an **odd** function $f(x)$.
The domain of $f(x)$ is $(-\infty, \infty)$ (all real numbers).

x	2	3	4	5
$f(x)$	-3	-1	-1	1

a. [3 points] Find the following values of f, or write NEI if there is "not enough information" to find the value.

(i) $f(-2) =$ **3**

(ii) $f(1) =$ **NEI**

(iii) $f(0) =$ **0**

Solution:

(i) Because $f(x)$ is odd, we know that $f(-2) = -f(2)$. From the table, we know that this is 3.

(ii) The table doesn't tell us about $f(1)$ or $f(-1)$, so we don't have enough information to say what this value is.

(iii) Because $f(x)$ is odd, $f(0) = f(-0) = -f(0)$. The only way $f(0) = -f(0)$ is if $f(0) = 0$, so that is the value.

b. [2 points] Could f be an invertible function? Explain your answer.

The function f (*circle one*): COULD BE INVERTIBLE $\boxed{\text{COULD NOT BE INVERTIBLE}}$

Explanation:

Solution: $f(x)$ could not be invertible. Because $f(3) = f(4) = -1$. Because two different inputs produce the same output, the function is not invertible.

c. [4 points] Recall that $f(x)$ is an odd function. For each of the following functions, decide whether it is even, odd, neither, or if there is not enough information (NEI) to tell. *No explanation needed.*

(i) The function $g(x) = x^3 f(x)$ is (*circle all that apply*):

ODD $\boxed{\text{EVEN}}$ NEITHER NEI

(ii) The function $h(x) = x^2 + f(x)$ is (*circle all that apply*):

ODD EVEN $\boxed{\text{NEITHER}}$ NEI

Solution:

(i) First note that $(-x)^3 = -x^3$. Then since $f(x)$ is odd,

$$g(-x) = (-x)^3 f(-x) = (-x^3)(-f(x)) = x^3 f(x) = g(x)$$

Thus $g(x)$ is even.

(ii) We can test one of the values of the table along with its negative, and see that $h(x)$ is neither even nor odd. For example, $h(2) = 4 - 3 = 1$, while $h(-2) = 4 + 3 = 7$. Since 1 and 7 are not equal or negatives of each other, $h(x)$ is neither even nor odd.

d. [1 point] Suppose it is also true that: $\lim_{x\to\infty} f(x) = 5$. Use this information to find $\lim_{x\to-\infty} f(x)$, or write NEI if there is not enough information to find the limit.

Solution: We know that $f(x)$ approaches 5 when x gets very large. Because $f(x)$ is odd, as x gets very negative, it will approach the opposite value of -5.

$\lim_{x\to-\infty} f(x) =$ -5

2. [10 points] The Executive Director of the Go Blue Zoo is managing the zoo's finances. She tracks the following functions:

- $O(n)$ is the number of pounds of fish per day used to feed n otters.
- $P(m)$ is the number of pounds of fish per day used to feed m penguins.
- $F(s)$ is the cost, in dollars, of s pounds of fish.

a. [6 points] For each of the following, describe the meaning of the expression or equation in the director's context, or explain why the expression or equation doesn't make sense.

(i) $O(6) = 70$

Solution: The zoo uses 70 pounds of fish per day to feed 6 otters.

(ii) $O^{-1}(P(8))$

Solution: The number of otters that can be fed with the same amount of fish used to feed 8 penguins.

(iii) $P(10) + F(10)$

Solution: This expression doesn't make sense because it doesn't make sense to add pounds of fish to dollars.

b. [4 points] For each of the following statements or quantities, write an equivalent mathematical expression or equation, or explain why there isn't enough information to do so.

(i) To feed its 5 otters and 8 penguins, the zoo uses 86 pounds of fish per day.

Expression or Equation: $O(5) + P(8) = 86$

(ii) The daily cost, in dollars, of feeding 5 otters.

Expression or Equation: $F(O(5))$

3. [11 points] The Go Blue Zoo's power has gone out and it is cold outside! The indoor frog exhibit is typically kept warm, but it is now getting colder. The exhibit temperature, in °F, t hours after the power goes out is given by:

$$E(t) = 30 + we^{kt}, \quad \text{where } w \text{ and } k \text{ are constants.}$$

a. [3 points] When the power first went out, the temperature of the frog exhibit was $75°F$, but after 4 hours the temperature is 68°F. Find the values of w and k.
Show all work. Give your answers in exact form, or accurate to two decimal places.

Solution: The first fact tells us that $E(0) = 75$, so

$$75 = 30 + we^(k \cdot 0) = 30 + w$$

Solving for w shows us $w = 45$.
The second fact tells us that $E(4) = 68$, so

$$68 = 30 + we^(k \cdot 1) = 30 + we^k = 30 + 45e^k$$

Solving for e^k shows us $e^k = 38/45$. Then, taking the natural logarithm of both sides, we get

$$k = ln(38/45).$$

$w =$ ________________ $k =$ ________________

b. [2 points] What is $\lim_{t\to\infty} E(t)$? Explain what this number means in the context of the problem.

Solution: Because $32/38 < 1$, $ln(32/38) < 0$, and the exponential expression $e^(ln(32/38)t)$ goes to 0 as $t \to \infty$. Thus $B(t)$ goes to 30 as $t \to \infty$. This is the temperature the frog exhibit will approach in the long run, which is the temperature of the environment outside.

$\lim_{t\to\infty} E(t) =$ __________

Meaning:

c. [4 points] Last summer the Go Blue Zoo also had a power outage. This time it was hot outside and the refrigerator for storing the tiger's food started warming up. After t hours, the temperature inside the refrigerator, in $°F$, was given by

$$R(t) = 75 - 38e^{-0.03t}$$

Due to safety concerns, food must be thrown out if the temperature inside a refrigerator rises above 40°F. How long could the power outage last without having to throw out the tiger's food? *Show all work. Give your answer in exact form, or accurate to two decimal places.*

Solution: We want to find the value of t such that

$$R(t) = 75 - 38e^{-0.03t} = 40$$

Isolating the exponential term, we get

$$35/38 = e^{-0.03t}$$

Then, taking the natural logarithm of both sides, we get

$$\ln(35/38) = -0.03t$$

so $t = \frac{\ln(35/38)}{-0.03}$.

____________________ hours

This problem continues on the next page.

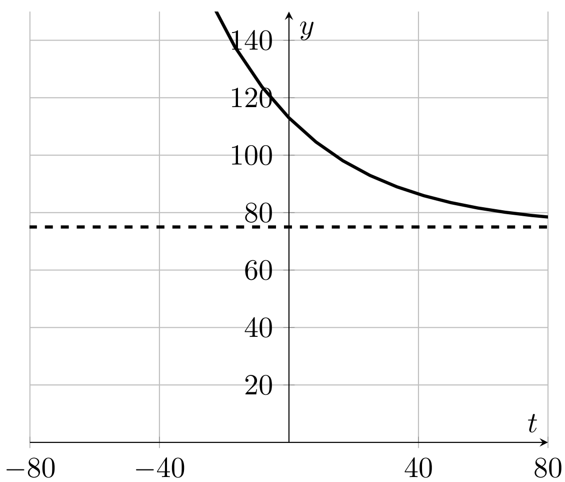
d. [2 points] Recall that we're considering the function

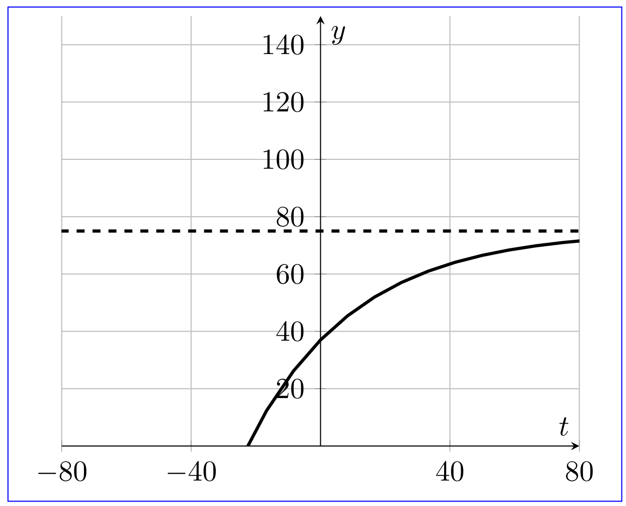
$$R(t) = 75 - 38e^{-0.03t}$$

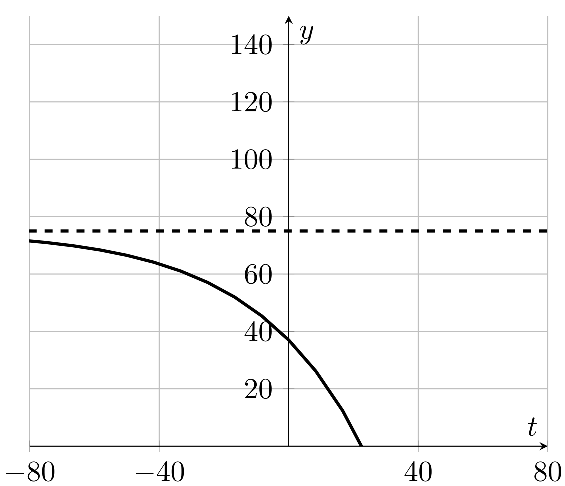
which tracks the temperature inside a refrigerator, in $^\circ F$, t hours after the power goes out on a hot day.

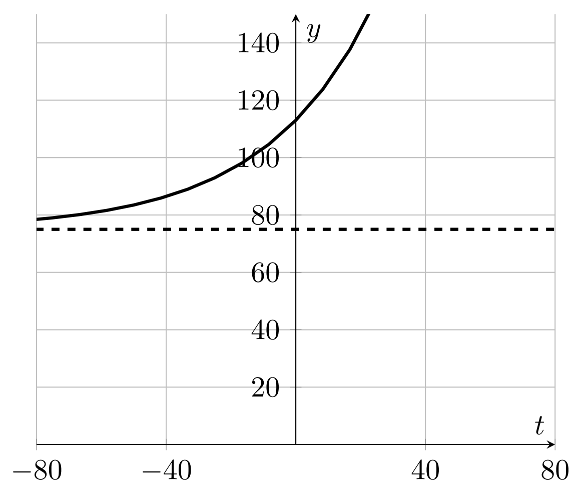
The domain of that function in *this* context would be $t \geq 0$; however, to help you make sense of meaningful features, the graphs below are shown on a larger domain.

Which of the four graphs show below could be the graph of $R(t)$? (*Circle one*)





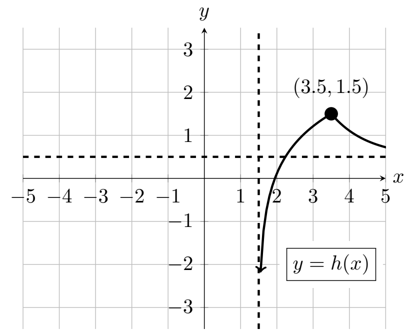




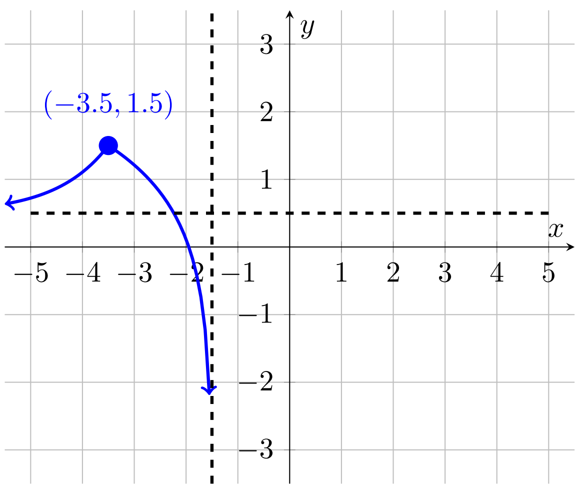
4. [11 points]

a. [6 points]

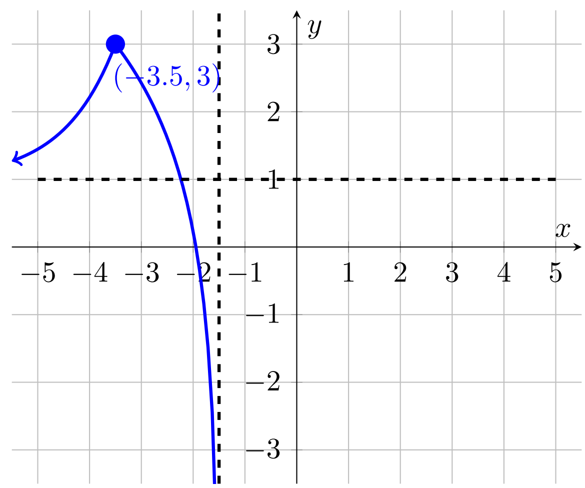
To the right is a graph of a function $h(x)$. The graph of $h(x)$ goes through the corner point $(3.5, 1.5)$, has a horizontal asymptote at $y = 0.5$, and a vertical asymptote at $x = 1.5$. We will apply a sequence of transformations to the graph of $h(x)$. For each subsequent transformation, sketch the intermediate graphical result on the given set of axes. **For each step, clearly denote any asymptotes with a dotted line, and label coordinates of the graph's corner point.**



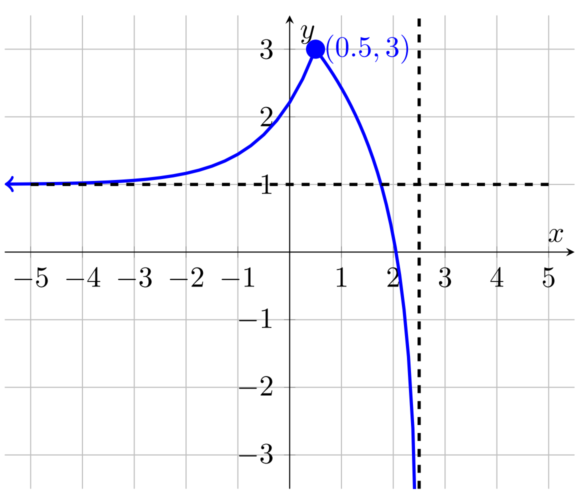
Step 1: Reflect the graph of $h(x)$ over the y-axis.



Step 2: Stretch the graph from Step 1 vertically by a factor of 2.



Step 3: Shift the graph from Step 2 right by 4.



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b. [3 points] Consider the function represented by the final graph in Step 3 and call it $g(x)$. Give a formula for $g(x)$ in terms of the original function $h(x)$.

$g(x) =$ $2h(-(x-4))$

c. [2 points] Use the final coordinates of your corner point in the graph of $g(x)$ (that is, Step 3) to check if your formula in part (b) is correct. *You can get points for this part of the problem even if your formula above is incorrect and you figure that out in this step, but don't know how to correct your formula.*

Solution:
Let's check that $g(0.5) = 3$, as is indicated by the corner point in our final graph. We can do that by plugging in 0.5 and going from there:

$$\begin{aligned} g(0.5) &= 2h(-(0.5-4)), \text{ using our proposed formula from part (b)} \\ &= 2h(-(-3.5)) \\ &= 2h(3.5) \\ &= 2(1.5), \text{ since our original corner point told us that } h(3.5) = 1.5 \\ &= 3 \end{aligned}$$

By plugging in 0.5 we see that our output is 3, which is also what we got in our final graph. This is good evidence that our formula is, indeed, correct!

5. [6 points] For fun, Booboo the chimpanzee likes to climb up a pole in the Ape House at the Go Blue Zoo and drop his doll Deedee. The function $T(h)$ gives the number of seconds it takes for Deedee to hit the ground when Booboo drops her from h feet above ground level. Write an expression for each of the following new functions, in terms of the function T:

a. [2 points] $R(h)$ is the time it takes, *in minutes*, for Deedee to hit the ground when dropped from h feet above ground level. (Note that 1 minute is equivalent to 60 seconds.)

$R(h) =$ $T(h)/60$

Solution: We need to take our output, which is in seconds, and convert it to minutes by dividing by 60.

b. [2 points] $F(y)$ is the time it takes, in seconds, for Deedee to hit the ground when dropped from y *yards* above ground level. (Note that 1 yard is equivalent to 3 feet.)

$F(y) =$ $T(3y)$

Solution: We need to take our new input units, which are yards, and convert them back to feet by multiplying by 3 before inputting into T.

c. [2 points] There is a small platform mounted on the pole 8 feet up from the ground. Sometimes Deedee lands there instead of the ground. The function $P(h)$ gives the time, in seconds, for Deedee to hit the platform when dropped from h feet above ground level. If we want to express $P(h)$ in terms of the function T, circle the best option below.

$P(h) = T(h+8)$ $\boxed{P(h) = T(h-8)}$ $P(h) = T(h) + 8$ $P(h) = T(h) - 8$

Solution: The platform 8 feet up is decreasing the total height the doll falls by 8 feet; so it's as if Booboo dropped the doll from 8 feet lower to being with.

6. [7 points] The Go Blue Zoo breeds its own insect populations for feeding some of the birds and reptiles. However, a virus infected the cricket breeding program on January 1, 2023. The cricket population started at 1,000,000 crickets, but decreased by 15% per week once the colony was infected.

a. [3 points] How many weeks did it take for the population to decrease by 60%? *Show all work. Give your answer in exact form, or accurate to at least two decimal places.*

Solution:
We need to solve the following equation for t:

$$400,000 = 1,000,000(0.85)^t$$

This simplifies to:

$$0.4 = 0.85^t$$

Taking log of both sides we get

$$\log(0.4) = \log(0.85^t)$$

And then applying log rules we get:

$$\log(0.4) = t\log(0.85)$$

Finally, isolating t using division we get:

$$\frac{\log(0.4)}{\log(0.85)} = t$$

$\frac{\log(0.4)}{\log(0.85)}$ weeks

b. [2 points] By what percentage did the cricket population decrease for each *day*? *Show all work. Give your answer in exact form, or accurate to at least two decimal places.*

Solution: One way to think about this problem is to think about solving for an unknown daily decay factor d that satisfies $d^7 = 0.85$. That is, a daily decay factor, when applied seven days in a row, should be equivalent to 0.85. If we solve for d we get

$$d = 0.85^{\frac{1}{7}}$$

But that tells us the daily decay *factor*. To find out the percent decease per day we need to compute $100 \times \left(1 - 0.85^{\frac{1}{7}}\right)$.
Note that it's tempting to use the known formula $r = b - 1$. That would, correctly, give us $100 \times \left(0.85^{\frac{1}{7}} - 1\right)$ as a precent *change*. However, that would give us the rate as a *negative* percent (because it's decaying). That's already captured in the fact that we're using the word "decay", so we want to make sure to give that as a positive number. That is, we wouldn't say something "decays by -5% daily".

Decreases by $100 \times \left(1 - 0.85^{\frac{1}{7}}\right)$ % per day.

c. [2 points] If the population instead decayed at a *continuous* rate of 15% per week, by what *non-continuous* percentage would it decrease in one week? *Show all work. Give your answer in exact form, or accurate to at least two decimal places.*

Solution: Similar to part (b), we need to find a decay factor, and then convert that to a percent change. In this case, if our continuous decay rate is 15%, then our growth (or decay) *factor* is $b = e^{-0.15}$. So our decay *rate* is

$$100 \times (1 - e^{-0.15})$$

Decreases by $\underline{\quad 100 \times (1 - e^{-0.15}) \quad}$ % per week.

7. [5 points]

a. [3 points] Let k, j be positive constants with $k > j > 0$. If any of the following values are undefined, write them in the "Undefined:" blank below. Then sort the remaining values from least to greatest.

0 $\log(-k)$ $\log(k/j)$ $\log(j/k)$ $\log(0)$

Undefined: 0**,**$\log(-k)$

Remaining quantities ordered from LEAST to GREATEST:

$\log(j/k) < 0 < \log(k/j)$

b. [2 points] If $\log(ab) = 2$, then...

Give a value for each of the following, or write NEI *if there is not enough information to answer.*

$ab =$ $10^2 = 100$

$\log(a) \cdot \log(b) =$ **NEI**