Math 105 — Final Exam — April 24, 2023

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. Use a pencil for "bubble-in" questions so that you can easily erase your answer if you change your mind.
4. This exam has 12 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
5. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
6. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
7. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
8. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
9. You must use the methods learned in this course to solve all problems.
10. You are allowed notes written on two sides of a $3'' \times 5''$ note card and one scientific calculator that does not have graphing or internet capabilities.
11. If you use a graph or table to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how the graph or table gives the answer.
12. Include units in your answer where that is appropriate.
13. Problems may ask for answers in *exact form* or in *decimal form*. Recall that $\sqrt{2} + \cos(3)$ is in exact form and $x = 0.424$ would be the same answer expressed in decimal form.
14. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is <u>not</u> permitted.

Problem	Points	Score
1	8	
2	7	
3	8	
4	7	

Problem	Points	Score
5	5	
6	10	
7	10	
8	5	
Total	60	

1. [8 points]

a. [3 points] Em, an employee at the *Math-tas-tíque Dog Boutique*, earns \$750 per week in salary and earns an additional 5% of her total sales that week (her *commission*). Write a formula for $M(x)$, the amount, in dollars, Em will earn in a week in which she is responsible for $\$x$ in sales.

Solution: We know that Em will make \$750 plus 0.05 times whatever the value of her sales were (x). So we get the linear function below.

$M(x) =$ $750 + 0.05x$

b. [3 points] Compute the value of $M^{-1}(1000)$ and describe its meaning in the context of the problem.
Show all work. Give your final answer in decimal form, NOT exact form.

Solution: Algebraically-speaking, $M^{-1}(1000)$ means the input x that will give us $M(x) = 1000$. So, using the formula above, we need to find x such that:

$$750 + 0.05x = 1000$$

Solving this:

$$0.05x = 250$$

$$x = 5000$$

$M^{-1}(1000) =$ **5000**

Meaning:

Solution: In order to make a salary of \$1000 in one week, Em needs to be responsible for \$5000 in sales.

c. [2 points] Let $R(w)$ be the function giving the dollar amount of Em's sales in the wth week of 2023. Choose the best description of the meaning of $M(R(23))$ from the choices below.

- ◯ A. The week in which Em makes \$23 in commission.
- ◯ B. The amount of commission Em makes in the 23rd week of 2023.
- ◯ C. The total amount Em gets paid in 2023.
- ◯ D. The total amount Em gets paid in in the 23rd week of 2023.
- ◯ E. This doesn't make sense because we cannot plug a number of weeks into the function M.

2. [7 points]

a. [4 points] A population of fleas takes residence at the nearby *I-Love-Functions Dog Hotel* (oh no!) and the population grows exponentially for the first couple of days. At $t = 2$ hours after the infestation started, the population is 1000 fleas. By $t = 6$ hours after it started, the population is 2000 fleas. Write a formula for $P(t)$, the number of fleas t hours after the infestation started.
Show all work. Your final formula should include parameters in their EXACT form.

Solution: We know points on our function: $P(2) = 1000$ and $P(6) = 2000$. We also know that P is, for a while at least, an exponential function, so of the form: $P(t) = ab^t$, where a and b as as-of-yet unknown parameters. We can use the two point we know to set up two equations with two unknown parameters a, b:

$$2000 = a \cdot b^6$$

$$1000 = a \cdot b^2$$

One way to work with these equations and solve for one of the paramaters is to divide one equation by the other. Doing this we get:

$$2 = b^4$$

So $b = 2^{\frac{1}{4}}$. We can plug this back into either equation to solve for the value of a:

$$1000 = a \cdot (2^{\frac{1}{4}})^2$$

$$1000 = a \cdot 2^{\frac{1}{2}} = a\sqrt{2}$$

$$a = \frac{1000}{\sqrt{2}}$$

Putting these values back in for the parameters of $P(t)$ we get the final formula below.

$P(t) =$ $\frac{1000}{\sqrt{2}}(2^{\frac{1}{4}})^t$

b. [3 points] *Last* year a population of fleas also took up residence at the hotel and their population, as a function of hours since their arrival, was given by:

$$Q(t) = 500(1.22^t)$$

By what percent did *this* population increase each hour?

22 %

How long did it take for their initial population to triple?
Show all work. Give your final answer in decimal form, NOT exact form.

Solution: We are trying to find the value of t such that:$1500 = 500(1.22^t)$
We can solve this as follows:

$$\begin{aligned} 1500 &= 500(1.22^t) \\ 3 &= 1.22^t \\ \ln(3) &= \ln(1.22^t) \\ \ln(3) &= t\ln(1.22) \\ \ln(3)/\ln(1.22) &= t \\ 5.5248 &\approx t \end{aligned}$$

5.5248 hours

3. [8 points] Traditionally, it has been assumed that a D year-old dog is the same biological age as a $7D$ year-old human. So a 3 year-old dog (in actual years) has aged as much as a 21 year-old human.

However, scientists have found a new aging formula for Labrador retrievers that takes specific biological aging markers into account. The new formula claims that a D year-old Labrador retriever (in actual years) has aged as much as a human who is

$$H = f(D) = 15\ln(D) + 31 \text{ years old}$$

One strange thing about this formula they came up with is that it doesn't go through the point $(0, 0)$ as we'd expect it to. In fact, we can't plug in 0 to this formula at all!

a. [2 points] Explain in one sentence why we can't plug $D = 0$ into this formula.

Explanation:

Solution: The domain of $\ln(D)$ does not include 0, so we can't plug $D = 0$ into this formula. (This was enough explanation for full credit on the exam.)

(For further thought:) The reason the domain doesn't include 0 is because ln is the inverse of the exponential function, and 0 isn't in the *range* of e^H. That is, there is no value H such that $e^H = 0$.

b. [3 points] According to this formula, at what age (in real years) will a dog be biologically equivalent to a newborn baby ($H = 0$)?
Show all work. Give your final answer in decimal form, NOT exact form.

Solution: You would expect that a 0-year old (newborn) dog would be equivalent to a 0-year old human (newborn). However, as we saw above, the formula doesn't actually go through the point $(0, 0)$ as we'd expect. In this question, we need to find the value of D such that: $0 = 15\ln(D) + 31$. We can solve this algebraically:

$$\begin{aligned} 0 &= 15\ln(D) + 31 \\ -31 &= 15\ln(D) \\ \frac{-31}{15} &= \ln(D) \\ e^{\frac{-31}{15}} &= D \end{aligned}$$

Putting this in decimal form using a calculator, we get that $D \approx 0.1266$. One way to interpret this would be that when a dog is 0.1266 years old, it is biologically equivalent to a newborn human. (It makes sense that this number is small!)

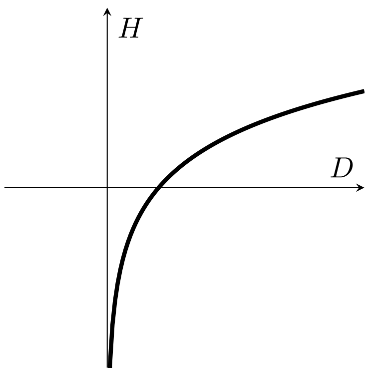
$D =$ **0.1266** years

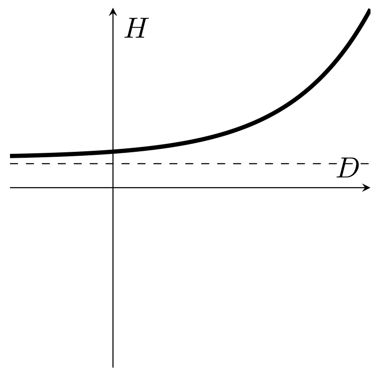
This problem continues on the next page.

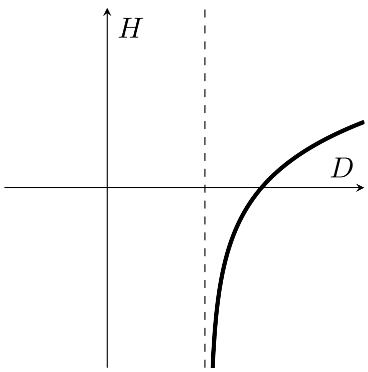
c. [3 points] Now considering the same function without its context: which of the graphs below could be the graph of

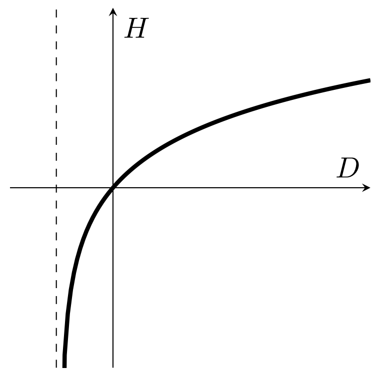
$$f(D) = 15\ln(D) + 31?$$

Circle the correct graph or NONE.









NONE OF THESE GRAPHS COULD REPRESENT THE FUNCTION $f(D)$.

Solution: The graph of $f(D)$ is related to the graph of $\ln(D)$. It is the original graph stretched vertically, then shifted up. Neither of these changes affect the position of the original vertical asymptote of $D = 0$, so the only possible graph is the one in the upper left, which is the same basic shape as the graph $\ln(D)$ function, with the vertical asymptote still at $D = 0$.
The upper right graph is a shift of e^D (not a ln function at all), and the two lower graphs are $\ln(D)$ functions that include a shift right or left.

4. [7 points] Dog owner Malik recently bought an Extra-High-Flying Ball™ at the *Math-tas-tíque Dog Boutique* for his extra-high-jumping Jack Rusell Terrier.

On one particular throw, the ball's height, in feet, is given by:

$$h(t) = -16(t + \tfrac{1}{8})(t - 3),$$

where t is the number of seconds after the ball left Malik's hand.

a. [2 points] At what height was the ball when it was released Malik's hand?
Show all work. Give your final answer in decimal form, NOT exact form.

Solution: Since t is the number of seconds since the ball left Malik's hand, we are looking to compute $h(0)$. We get

$$-16(0 + \frac{1}{8})(0 - 3) = 6$$

height: **6** feet

b. [3 points] What is the maximum height the ball reached and at what time did it reach that height?
Show all work. Give your final answer in decimal form, NOT exact form.

Solution: We are essentially trying to find the coordinates (t, h) of the vertex of this parabola. One way to do this it to utilize the natural symmetry of a parabola. A parabola is symmetric across the vertical line going through its vertex. Since the two zeros will be symmetrical about that line of symmetry, this means that the t-coordinate of the vertex (or the time of maximum height) will be the midpoint of the two zeros.
From the factored form of the function given to us, we can see that the zeros are at $t = -\frac{1}{8}, 3$. The midpoint of these two values can be found using the same method we use to find an average of two numbers:

$$\frac{3 + \left(-\frac{1}{8}\right)}{2} = 1.4375$$

To find the actual maximum height acheived we need to plug that t-value into the original function $h(t)$:

$$-16(1.4375 + \tfrac{1}{8})(1.4375 - 3) = 39.0625$$

height: **39.0625** feet

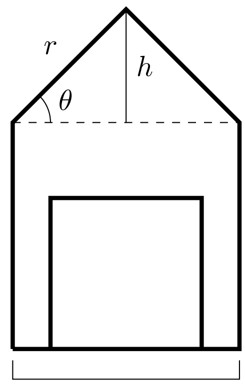
time: **1.4375** seconds

c. [2 points] Assuming the ball wasn't caught on its way down, how many seconds, total, was the ball in the air?

Solution: Since the ball is in the air starting at $t = 0$ until the time it hits the ground at $t = 3$ (one of the zeros of our function), it is in the air for 3 seconds total before hitting the ground.

time in the air: **3** seconds

5. [5 points] Another customer of the dog boutique is making a custom dog house. A sketch of their plans (not drawn to scale) is shown below:



a. [2 points] In order to for the snow to slide off, the slope of the roof should rise at least 4 inches vertically for each 12 inches in horizontal change. **If $\theta = 20°$ will the roof be steep enough for snow to slide off?** *Show all expressions that you calculate.*

Solution: There are many ways to approach this! The simplest is probably to recall that $\tan(\theta)$ gives the slope of the line making that angle with the x-axis. With that in mind, we can use a calculator to compute $\tan(20°) = 0.36397$. Since the slope we need to let snow run off is $\frac{1}{3}$ and $0.36397 > 0.333...$, the $20°$ angle shed roof is steep enough.

In a slight variation of the above method, some students used $\tan(20°)$ to find the value of h in the diagram is 0.1274 m and then similarly found that:

$$0.1274/0.35 > 0.333....$$

There were many other methods possible to approach this problem. Some used a conversion from meters to inches, but this was not at all required or necessary to approach this problem.

(Circle one) **[YES]** NO NOT ENOUGH INFORMATION

b. [3 points] The dog owner decides to make $\theta = 22°$. If the overall width of the front piece shown is 0.7 meters, what will be the measurement of r shown in the diagram?
Show all work. Give your final answer in decimal form, NOT exact form.

Solution: We know that $\cos(22°) = 0.35/r$. Solving for r we get

$$r = 0.35/\cos(22°) \approx 0.377 \text{ meters}$$

$r =$ **0.377** meters

6. [10 points] Color in the blank circle for **all possible** correct choices. *Remember to use pencil so that you can erase your answers if you change your mind!*

a. [2 points] A graph goes through the points $(1, 2)$ and $(-1, 6)$.

This graph *could* represent a(n) ____________ function.

- ◯ linear
- ◯ exponential
- ◯ periodic
- ◯ odd
- ◯ NONE OF THE ABOVE

b. [2 points] A graph goes through the points $(2, 4)$ and $(2, 10)$.

This graph could represent a(n) ____________ function.

- ◯ linear
- ◯ exponential
- ◯ periodic
- ◯ odd
- ◯ NONE OF THE ABOVE

This problem continues on the next page.

c. [2 points] $f(x) = 4(x-2) + 3x + 8$.

$f(x)$ is a(n) ____________ function.

- ◯ linear
- ◯ exponential
- ◯ periodic
- ◯ odd
- ◯ NONE OF THE ABOVE

d. [2 points] $g(x) = e^{3(x-4)}$.

$g(x)$ is a(n) ____________ function.

- ◯ linear
- ◯ exponential
- ◯ periodic
- ◯ odd
- ◯ NONE OF THE ABOVE

e. [2 points] $h(x) = \frac{2}{3}\sin(4x)$

$h(x)$ is a(n) ____________ function.

- ◯ linear
- ◯ exponential
- ◯ periodic
- ◯ odd
- ◯ NONE OF THE ABOVE

7. [10 points] We start with the function $f(x) = \cos x$ and perform the following transformations to its graph:

(i) Stretch it vertically by a factor of 2.5

(ii) Compress it horizontally by a factor of $\frac{1}{3}$

(iii) Shift it vertically, down by 1

(iv) Shift it horizontally right by π.

a. [4 points] Call the function represented by the new graph $g(x)$. What is a formula for this new function $g(x)$?

$g(x) =$ $2.5\cos(3(x-\pi))-1$

b. [2 points] What is an equation for the midline of $g(x)$?

$y =$ -1

c. [2 points] What is the amplitude of $g(x)$?

Amplitude: 2.5

d. [2 points] What is the period of $g(x)$?

Period: $2\pi/3$

8. [5 points] The *I-Love-Functions Dog Hotel* has a one-of-a-kind Doggie Ferris Wheel for its residents to use on special occasions. The hotel residents board the Doggie Ferris Wheel at its lowest point, from a platform that is 5 feet high. The Doggie Ferris Wheel is 34 feet in diameter.

a. [3 points] If each full rotation rotation takes 1 minute, how high off of the ground is a dog when she is exactly 20 seconds into the ride?
Show all work (including any pictures). Give your final answer in decimal form, NOT exact form.

Solution: There were multiple ways to approach this problem! One way was to first model the height of a dog using a sinusoidal function. Since this function starts at its minimum value (when the dog boards), our simplest starting function will be $\cos(x)$. Then factoring in the amplitude, period, and shift, we end up with the following height formula as a function of time t in seconds:

$$h(t) = -17\cos\left(\frac{2\pi}{60}t\right) + 22$$

To compute the dogs height at 20 seconds, we can plug $t = 20$ into the above formula and get $h(20) = 30.5$.

We need to consider whether or calculator should be in radians or degrees when we do this. The coefficient $2\pi/60$ was implicitly assuming we were starting with a cos function with period 2π, so our calculator should be in radians to use this formula. If we didn't realize that and it was in degrees, then we'd get 5.011 feet. However, since they're boarding at 5 feet and going all the way up to 39 feet, it doesn't make sense that 20 seconds into a 60-second rotation would have the dog just slightly above the boarding height—so we could catch our own mistake this way.

There is a totally different way of answer this problem. Consider a Ferris Wheel with center $(0, 22)$ and let's figure out where on the Ferris Wheel we'd be at 20 seconds. Since 20 seconds is a third of a 60-second rotation, it would put the dog at $360°/3 = 120°$ from the boarding point, so at 30° above the horizontal point ("3 o'clock position"). The height at this angle above the "3 o'clock position" will be $\sin(30°) \cdot 17 = \frac{1}{2} \cdot 17 = 8.5$ feet. If we add this to a starting height at the 3 o'clock position of 22 feet, we get the same height as the method above: 30.5 feet.

Height: **30.5** feet

b. [2 points] What length of the Doggie Ferris Wheel's arc is traversed by a passenger dog in 47 seconds of riding?
Show all work (including any pictures). Give your final answer in decimal form, NOT exact form.

Solution: The total circumference of the Ferris Wheel is $2\pi \cdot 17 = 34\pi \approx 106.81$ feet. The dog is riding for $\frac{47}{60}$ of a full rotation, so a total of $\frac{47}{60} \cdot 34\pi = 83.671$ feet.

Length of arc: **83.671** feet