Math 105 — First Midterm — February 19, 2024

EXAM SOLUTIONS

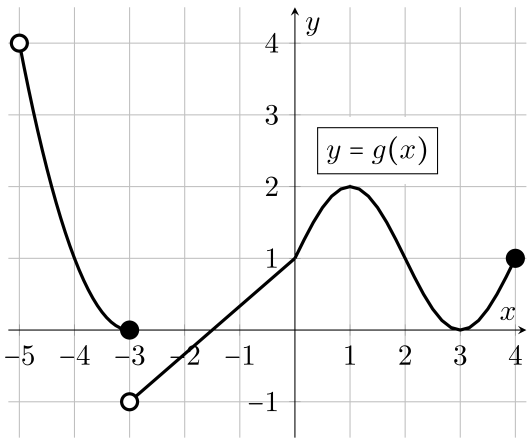
1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. There are 7 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratch-work. Clearly identify any of this work that you would like to have graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are allowed notes written on two sides of a $3'' \times 5''$ note card and one scientific calculator that does not have graphing or internet capabilities.
10. Include units in your answer where that is appropriate.
11. Problems may ask for answers in *exact form* or in *decimal form*. Recall that $\sqrt{2} + \cos(3)$ is in exact form and 0.424 would be the same answer expressed in decimal form.
12. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

Problem	Points	Score
1	12	
2	9	
3	6	
4	8	

Problem	Points	Score
5	6	
6	11	
7	8	
Total	60	

1. [12 points] Consider the following functions:

- $f(x) = 2(x - 1) - 5$
- $g(x)$ is given by the graph below.
- Some values of the function $h(x)$, which has an inverse function h^{-1}, are given in the table below.



x	-2	-1	0	1	2
$h(x)$	5	2	-1	-10	0

a. [2 points] Over which of the following intervals does $g(x)$ appear to be concave up on the **entire interval**? *Circle all that apply.*

[$(-5,-3]$] $(-3,0)$ $(1,4]$ [$(2,4]$] NONE

b. [2 points] Over which of the following intervals does $g(x)$ appear to be increasing on the **entire interval**? *Circle all that apply.*

$(-5,-3]$ [$(-3,1)$] $(2,4]$ [$(3,4]$] NONE

c. [2 points] Give a formula for a linear function $w(x)$ whose graph is perpendicular to the graph of $f(x)$ and goes through the point $(3,-2)$.

$w(x) =$ $-\frac{1}{2}(x-3) - 2$

d. [6 points] Find the value of the following quantities, where possible. Write N/A if they cannot be determined or do not exist.

(i) $f^{-1}(9) =$ 8

(ii) $f(g(-3)) =$ -7

(iii) $h^{-1}(g(3)) =$ 2

(iv) If $w(x) = g(x-1) - 3$, $w(2) =$ -1

(v) All x such that $g(x) = 1$: $x =$ $-4, 0, 2, 4$

2. [9 points] The height of water in a cylindrical tank, as it drains out, is given by

$$H = h(t) = 4t^2 - 40t + 100,$$

where H is measured in centimeters and t is measured in minutes after a spigot is opened. The formula holds until the tank is emptied, after which, the height does not change anymore.

For your reference, the zeros of $y = ax^2 + bx + c$ *can be found by the formula:* $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

a. [2 points] How high is the water in the tank when the spigot is first opened? *Give your answer in exact form, or rounded to two decimal places. Include units.*

Solution: $h(0) = 4(0)^2 - 40(0) + 100 = 100$

Water height: 100 **centimeters**

b. [2 points] After how many minutes is the tank empty? *Show all work. Give your final answer in exact form or rounded to two decimals places.*

Solution: We need to find the first t-value for which $0 = 4t^2 - 40t + 100$.
We can first try factoring to solve:

$$0 = 4t^2 - 40t + 100$$
$$0 = 4(t^2 - 10t + 25)$$
$$0 = 4(t-5)(t-5)$$

Since this was factorable, we can see that the only solution is when $t = 5$ minutes.

5 minutes

c. [2 points] What is a reasonable domain and range for this function in the context of the problem? *Use inequality OR interval notation for your answer.*

Domain: $[0, 5]$ **Range:** $[0, 100]$

d. [3 points] How long does it take for the tank to be half as full as it started? *Show all work. Give your final answer rounded to two decimals places.*

Solution: We need to find out how long it takes for the water to reach 50cm. That is, we need to solve the following equation for t:

$$50 = 4t^2 - 40t + 100$$
$$0 = 4t^2 - 40t + 50$$
$$0 = 2t^2 - 20t + 25$$

We can use the quadratic formula to find all possible solutions:

$$t = \frac{20 \pm \sqrt{400 - 200}}{4}$$
$$t = \frac{20 \pm \sqrt{200}}{4}$$
$$t = \frac{20 \pm 10\sqrt{2}}{4}$$
$$t = 5 \pm \frac{5\sqrt{2}}{2}$$

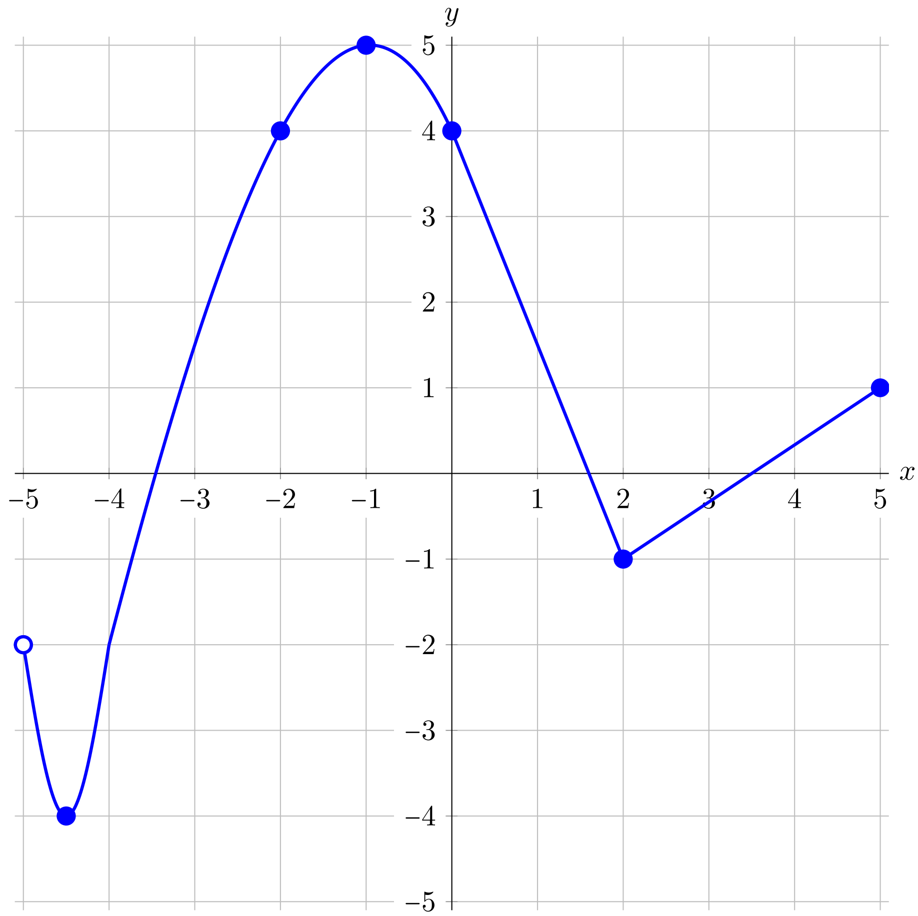
Only one of these solutions actually occurs before the tank is empty (during the range of this problem), and that is the smaller one. So our only solutions is $t = 5 - \frac{5\sqrt{2}}{2} \approx 1.464466$

$5 - \frac{5\sqrt{2}}{2} \approx 1.464466$ minutes

3. [6 points] On the axes below, sketch the graph of a **single** function $y = f(x)$ with all of the following properties:

- $f(x)$ has its vertical intercept at $y = 4$,
- the average rate of change of f between $x = -2$ and $x = 0$ is 0,
- $f(x)$ is concave down and increasing on the interval $-4 < x < -1$,
- $f(x)$ has a constant rate of change on $0 < x < 2$ with slope $-\frac{5}{2}$,
- the domain of $f(x)$ is $-5 < x \leq 5$,
- the range of $f(x)$ is $-4 \leq f(x) \leq 5$.

Solution: There are many possibilities for this graph! One is shown below. Some of the closed circles are to emphasize certain included points meeting the question requirements.



4. [8 points] For a certain computer, $P(f)$ measures the amount of power the computer consumes in Watts (W) as a **linear** function of the frequency, f, which is measured in Gigahertz (GHz). At the frequency $f = 0.8$ GHz the computer consumes 8 W of power; at the frequency $f = 3.4$ GHz the computer consumes 125 W of power.

a. [3 points] Find the slope of $P(f)$ and give its units. *Show all work. Give your answer rounded to at least two decimal places.*

Solution: $\frac{\text{change in output}}{\text{change in input}} = \frac{125\text{W}-8\text{W}}{3.4\text{GHz}-0.8\text{GHz}} = \frac{117\text{W}}{2.6\text{GHz}} = 45\text{W/GHz}$

45 W/GHz

b. [2 points] Suppose we open a new application on the computer and the frequency increases by 2.8 GHz. By how many watts (W) did the power consumption, $P(f)$, increase? *Show all work. Give your answer in exact form or rounded to at least two decimal places.*

Solution: $45\,\text{W/GHz} \times 2.8\,\text{GHz} = 126\,\text{W}$

126 Watts

c. [3 points] Find a formula for $P(f)$. *Show all work. Express all constants in exact form or rounded to at least two decimal places.*

Solution: One of the fastest (and more error-free!) ways to answer this question is using point-slope form. Since we know the slope is 45 W/GHz and we know one point on the graph is $(0.8, 8)$, we can find the formula for the line as shown below:

$P(f) =$ $45(f - 0.8) + 8$

5. [6 points] Let $W(d)$ be the probability that the great soccer player Pelénomial scores when he takes a shot d yards away from the goal line. Some values of $W(d)$ are given in the table below.

d	0	6	12	18
$W(d)$	0.94	0.4831	0.2483	

a. [2 points] Is $W(d)$ modeled better by a linear function or by an exponential function? *To receive credit, you must test both models and show all work.*

Solution: To check if the function is linear, we can calculate average rates of change:

$$\frac{0.4831 - 0.94}{6} = -0.07615$$

$$\frac{0.2483 - 0.4831}{6} = -0.0391\overline{3}$$

Since these average rates of change are not equal (or even close to equal), the function $W(d)$ does not seem to be well modeled by a linear function.
To check if $W(d)$ is approximately exponential, we can check if successive ratios outputs (from evenly spaced inputs) are equal (or close to equal). Since our inputs are evenly spaced, we need only check the following ratios:

$$\frac{0.2483}{0.4831} \approx 0.514$$

$$\frac{0.4831}{0.94} \approx 0.514$$

Since they are approximately equal (and the slopes calculated above are not at all close), we can conclude that the exponential function in a better fit.

(Circle one) LINEAR **EXPONENTIAL**

b. [2 points] If you said above that $W(d)$ was linear, find its slope. If you said above that $W(d)$ was exponential, find its approximate growth factor. *Show all work or point to relevant work above. Give your answer rounded to two decimal places.*

Solution: Let us call the growth factor b. Then from the above table and work we know that: $b^6 \approx 0.514$. So $b \approx 0.514^{\frac{1}{6}} \approx 0.895$

SLOPE (if linear) / GROWTH FACTOR (if exponential): 0.895

c. [2 points] Use your work above to compute the probability that Pelénomial scores when he takes a shot 18 yards away from the goal line. *Show all work. Give your answer in exact form, or rounded to two decimal places.*

Solution: One way to compute this is to notice that for every 6 yds, we found above that the probability is multiplied by 0.514. So the probability at 18 yards should be 0.514 times the probability at 12 yards, or $0.2483 \times 0.514 \approx 0.1276$.
We could also approach this by using the fact that our growth factor above tells us that for each additional yard, the probability is multiplied by 0.895. So to find the probability at 18 yards, we could start with the probability at 0 yards and get: $0.94 \times (0.895)^{18} \approx 0.1276$

Answer: ≈ 0.1276

6. [11 points] The following problem parts are not related.

a. [2 points] A ball is thrown up in the air from a platform and its height in meters above the ground is

$$H(t) = -4.9(t-0.9)^2 + 4.5,$$

where t is measured in seconds. What is the greatest height above the ground the ball reaches? And when does it reach that height?

Greatest height: **4.5** meters Time: **0.9** seconds

b. [4 points] Write a formula for a population of bacteria $P(t)$ that starts with a population of 10^5 and grows by 30% every day. The variable t is measured in days after the experiment starts.

$P(t) =$ $10^5 \cdot 1.3^t$

If $E(p)$ is the rate at which is energy is given off, measured in joules/second, by p bacteria of this kind, what is the meaning of the following equation?

$$E(P(2)) = 0.3$$

Meaning: Two days after the experiment starts, the bacteria population is giving off 0.3 joules / second.

c. [5 points] A table of some values of the function $h(r)$ is given below:

r	-2	0	2	4
$h(r)$	-3	-1	10	5

Let $g(r) = h(r-1) + 3$.

To obtain the graph of $g(r)$, one must shift the graph of $h(r)$...

- ...vertically (CIRCLE ONE) [UP] DOWN by **3**
- ...horizontally (CIRCLE ONE) LEFT [RIGHT] by **1**

From the given information, we can deduce the coordinates of several points on the graph of $g(r)$. Give the coordinates of two such points:

Solution: We can take any of the points give in the table and translate their r-coordinate to the right 1 and their corresponding output up by 3. Two examples are given below:

$(1, 2)$ and $(3, 13)$

7. [8 points] Mx. Miserable's Maids (or, MMM) charges

- an \$18 "drive up" fee per house cleaning;
- \$40 per 1000 square feet for the first 2500 square feet;
- then \$32 per 1000 square feet beyond 2500.

For example, the cost to clean a 750 square foot house is \$18 + \$30 = \$48. Additionally, they will not take on any clients with houses greater than 5000 square feet.

a. [2 points] How much does MMM charge for cleaning a 3000 square foot house? *Show all work. Given your answer rounded to at least two decimal places.*

Solution: They will charge

- \$18 for the drive up fee;
- $\$40 \times 2.5 = \$100$ for the first 2500 square feet;
- and $\$32 \times 0.5 = \$16$ for the remaining 500 square feet.

In total, that is $\$18 + \$100 + \$16 = \$134$

134 dollars

b. [6 points] Let $C(a)$ be the price, in dollars, that MMM charges for a single visit to a house that is a **thousand square feet**. Find a piecewise-defined formula for $C(a)$:

Solution: For the first 2500 square feet, the rate is \$40 / 1000 sq. feet, plus a fixed cost of \$18. So the initial formula is $18 + 40a$, where a is the number of square feet *in thousands*. Since this rate is good up to 2500 square feet, this formula holds up to $a = 2.5$. There are several ways of approaching the solution to the second part of the formula. One thing we know is that the slope is 32, because we're told that the rate is \$32 / 1000 sq. feet, once we're above 2500 sq. feet. But we need to add some constant to this. We could set up slope-intercept form ($y = ma + b$) and use a point to solve for b. However, point-slope form is quite a bit easier. In this case, when $a = 2.5$, the overall cost $C(2.5)$ should be $18 + 2.5 \times 40 = 118$. This means our two pieces of our graph meet at the point $(2.5, 118)$. So we can use this point in our point-slope form and get $C(a) = 32(a - 2.5) + 118$. This rate will hold for areas between 2500 square feet and 5000 square feet, so for $2.5 \le a \le 5$.

$$C(a) = \begin{cases} 18 + 40a, & 0 \le a \le 2.5 \\ 32(a - 2.5) + 118, & 2.5 \le a \le 5 \end{cases}$$