Math 105 — Second Midterm — April 1, 2024

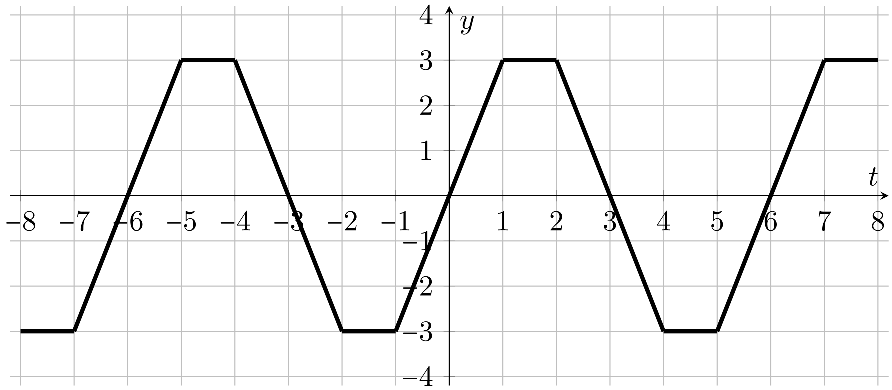
EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. There are 6 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratch-work. Clearly identify any of this work that you would like to have graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are allowed notes written on two sides of a $3'' \times 5''$ note card and one scientific calculator that does not have graphing or internet capabilities.
10. Include units in your answer where that is appropriate.
11. Problems may ask for answers in *exact form* or in *decimal form*. Recall that $\sqrt{2} + \cos(3)$ is in exact form and 0.424 would be the same answer expressed in decimal form.
12. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is <u>not</u> permitted.

Problem	Points	Score
1	12	
2	9	
3	11	

Problem	Points	Score
4	8	
5	10	
6	10	
Total	60	

1. [12 points] Below is a graph of periodic, odd function $h(t)$, with period 6:



a. [3 points] A **new** function $g(t) = -h(2t)$ ...

- ... has period: **3**
- ... is (CIRCLE ONE) $\boxed{\textbf{ODD}}$ **EVEN** **NEITHER**
- ... has amplitude: **3**

Solution: Because $g(t)$ is a horizontal compression of $h(t)$ by a factor of $1/2$, its period will be half as large, so 3.

When we horizontally stretch $h(t)$ and reflect it over the t-axis, it still retains its symmetry about the origin, so is still odd. We could also see this algebraically, using the fact that $h(t)$ is odd:

$$g(-t) = -h(-2t) = -(-h(2t)) = -g(t)$$

The amplitude of $h(t)$ is unchanged by the transformations, so the amplitude of $g(t)$ is still 3.

b. [3 points] Another **new** function $w(t) = 3h(t-1)$ ...

- ... has period: **6**
- ... is (CIRCLE ONE) **ODD** **EVEN** $\boxed{\textbf{NEITHER}}$
- ... has maximum value: **9**

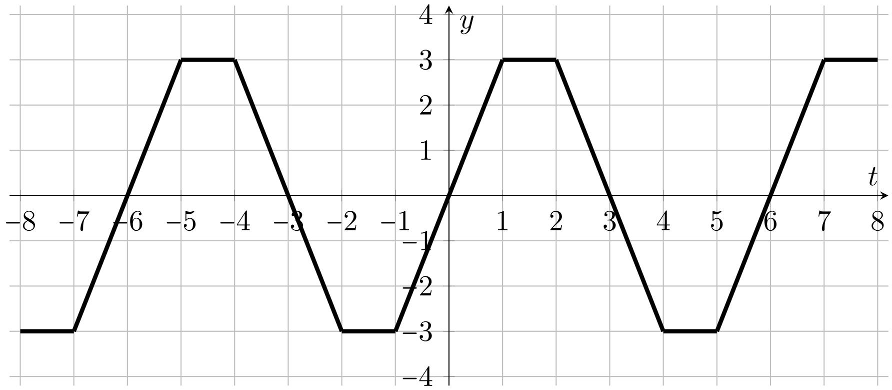
Solution: Because $w(t)$ has not been stretched or compressed horizontally, its period will remain the same: 6.

When we horizontally shift $h(t)$ right by 1, it loses its symmetry about the origin. It is neither odd nor even after that shift.

$w(t)$ is a vertical stretch of $h(t)$ by a factor of 3, so its amplitude is 3 times as large, so 9.

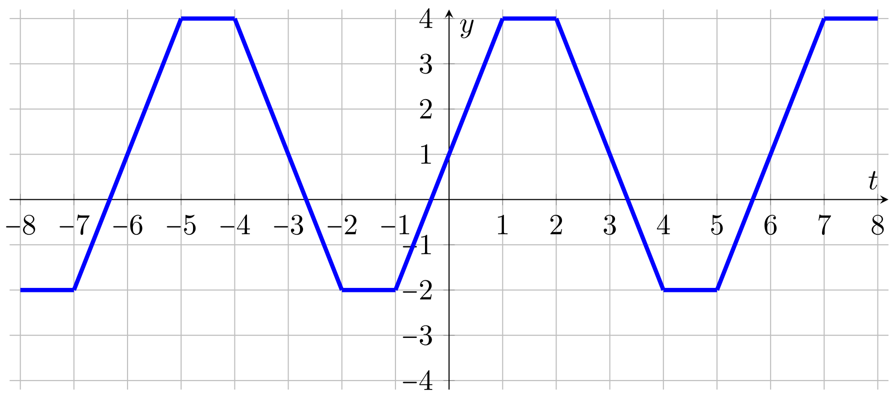
This problem continues on the next page.

The graph of $h(t)$ is reproduced here for your convenience.

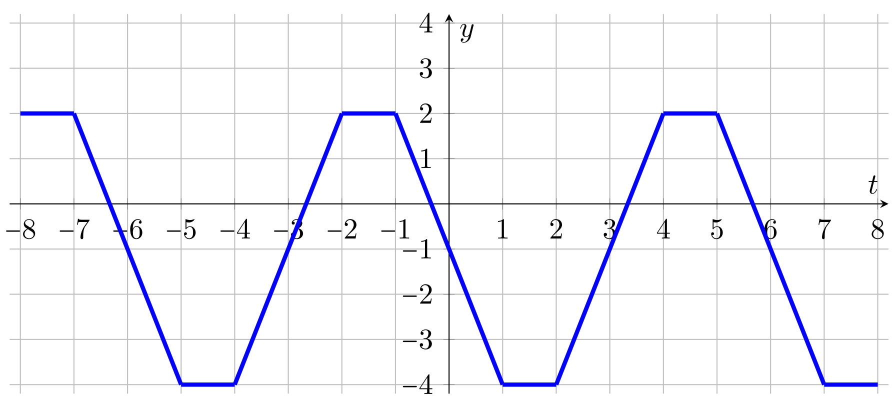


c. [6 points] Carry out the following sequence of transformations to the graph of $h(t)$. Draw each intermediate graph on the provided axes. *Clearly label at least three specific, known points in each graph.*

1. Shift the graph of $h(t)$ up by 1 unit.



2. Reflect the resulting graph over the t-axis.



Call the function in the final graph $k(t)$. What is a formula for $k(t)$ in terms of $h(t)$?

$k(t) =$ $-(h(t)+1)$ **or, equivalently,** $-h(t)-1$

2. [9 points] A group of scientists is modeling the transmission of light through different liquids. The functions below measure the brightness of the light, in lumens, at a depth of d cm below the surface of two different liquids: A and B.

$$A(d) = 45e^{-0.001d}$$
$$B(d) = 50e^{-0.001(2d-25)}$$

The functions $A(d)$ and $B(d)$ have a domain of $[0, \infty)$.

a. [1 point] How bright is the light at the surface of liquid B? *Express your answer in exact form, or rounded to at least two decimal places.*

Solution:
The brightness at the surface of liquid B is $B(0) = 50e^{-0.001(0-25)} = 50e^{0.025} \approx 51.27$ lumens.

$50e^{0.025} \approx 51.27$ lumens

b. [4 points] At what depth do the lights in the experiments with liquids A and B have the same brightness? *Show all work. Express your answer in exact form, or rounded to at least two decimal places.*

Solution:
To find where the brighntesses are equal, we need to solve $A(d) = B(d)$.

$$45e^{-0.001d} = 50e^{-0.001(2d-25)} \implies \ln(45) - 0.001d = \ln(50) - 0.001(2d - 25)$$
$$\implies 0.001d = \ln(50) - \ln(45) + 0.025$$
$$d = \frac{\ln(50) - \ln(45) + 0.25}{0.001} \text{ cm}$$
$$d \approx 130.36 \text{ cm.}$$

$\dfrac{\ln(50) - \ln(45) + 0.25}{0.001} \approx 130.36$ cm

c. [4 points] In a third experiment the scientists observe that the brightness of a light decreases by 10% for every 5 cm of depth below the surface of a liquid C. No matter the starting depth, how much deeper do you need to go to reduce the brightness by 25%? *Show all work. Express your answer in exact form, or rounded to at least two decimal places.*

Solution:
$$(0.9)^{d/5} = 0.75 \implies \frac{d}{5}\log(0.9) = \log(0.75) \implies d = \frac{5\log(0.75)}{\log(0.9)} \approx 13.65 \text{ cm.}$$

$\dfrac{5\log(0.75)}{\log(0.9)} \approx 13.65$ cm

3. [11 points] Asteroid Mining Co. hauls mineral-rich asteroids from across the solar system back to Earth's orbit for mining. The scientists at the company use the following functions to compute instructions for the space ship's crew.

- $S(p)$ is the amount of fuel, measured in liters, that is needed to move the ship p parsecs (a unit of distance).
- $F(s)$ is the amount of fuel, measured in liters, that is used when the engines run for s seconds.

Assume that $S(p)$ and $F(s)$ are both invertible.

a. [6 points] For each of the expressions below, give an interpretation or explain why the expression doesn't make sense.

- $S^{-1}(2.3 \times 10^8) = 0.2$

Solution:
The space ship needs 2.3×10^8 liters of fuel to travel 0.2 parsecs.

- $S(F(60))$

Solution:
This composition doesn't make sense. The output of F is liters of fuel, but the input of S is distance in parsecs.

- $F^{-1}(S(20))$

Solution:
The amount of time, in seconds, that the engines need to run for the ship to move 20 parsecs.

b. [2 points] Write an expression for the amount of fuel used, in liters, when the ship travels 2.8 parsecs and then runs the engines for an additional 30 seconds. Your answer may include any of S, F, S^{-1}, or F^{-1}.

$S(2.8) + F(30)$ liters

c. [3 points] Define the function $D(m)$ to be the distance, in parsecs, the ship moves when the engines run for m **minutes**. Write a formula for $D(m)$ in terms of F, S, F^{-1}, and/or S^{-1}.

Solution: The composition: time (seconds) → fuel (liters) → distance (parsecs) is $S^{-1}(F(s))$, where s is time measured in seconds. A minute is 60 seconds, so $s = 60m$.

$D(m) =$ $S^{-1}(F(60m))$

4. [8 points] The following parts are unrelated.

a. [4 points] The quantity of an intravenous drug in a patient's body, in mg, is given by $D(t) = 250(0.88)^t$, where t is the number of hours after the drug was administered.

What is the hourly decay rate of the drug? *Express in exact form, or rounded to at least 2 decimal places.*

Solution: We can see in the formula that the growth factor, b, is 0.88. This means that 88% of our drug remains after each hour, so it has decayed by 12%. This means our decay rate is 12%. We could also find this using $100 \times (1 - 0.88)$.

12%

What is the ***continuous*** hourly decay rate of the drug? *Express in exact form, or rounded to at least 2 decimal places.*

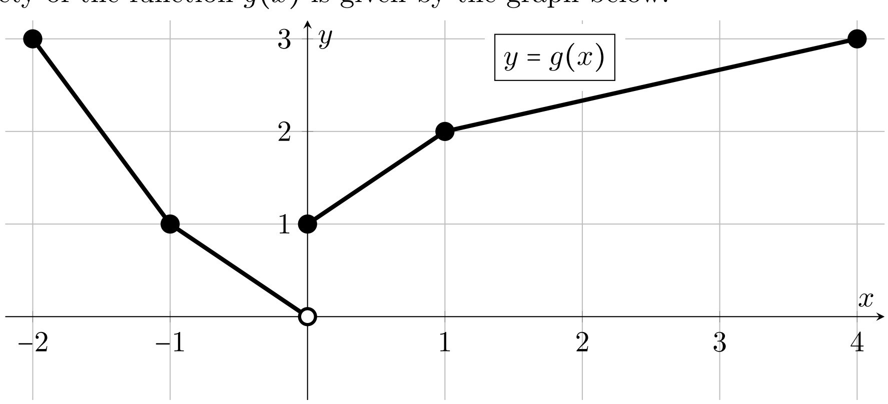
Solution: $e^k = 0.88$, so $k = \ln(0.88) \approx -0.12783$. This means that the continuous decay rate is approximately 12.783%.

12.783%

b. [4 points] The function $f(x)$ is given by the following formula:

$$f(x) = \ln(x) + 2$$

The entirety of the function $g(x)$ is given by the graph below.



Find the following values, or write NEI if there is "not enough information" to compute them. *Show all work.*

- $f^{-1}(g(0)) =$ $\frac{1}{e}$

Solution: First, we can use the graph to find that $g(0) = 1$. Thus we need to compute $f^{-1}(1)$. One way to do this is to solve the following for x:

$$\ln(x) + 2 = 1$$

That is, finding the value of x that gave an output of 1. We can solve that as follows:

$$\ln(x) = -1$$

$$e^{-1} = \frac{1}{e} = x$$

We could have also found a general formula for f^{-1} and evaluated it at 1. That general formula would turn out to be:

$$f^{-1}(y) = e^{y-2}$$

giving the same value of $f^{-1}(1) = e^{1-2} = e^{-1}$ as above.

- All x such that $g(g(x)) = 2$: $x =$ $-1, 0$

Solution: We want to know where the outer function $g(...)$ gives an output of 2. It gives an output of 2 exactly when its input is 1 or when its input is -1.5. That is, we need to know when the inner function (also $g(x)$ in this case!) will give us either 1 or -1.5. The number -1.5 is not in the range of $g(x)$, so we only need to solve for when $g(x) = 1$. This happens exactly when $x = -1, 0$. Let's plug those numbers back in to check that we get what we were hoping for:

$$g(g(-1)) = g(1) = 2 \checkmark$$

$$g(g(0)) = g(1) = 2 \checkmark$$

5. [10 points] The temperature T in a given room, measured in °F, after an air conditioner is turned on is given by $T = f(t) = 68 + 5e^{-0.02t}$, where t is measured in minutes.

a. [4 points] Find the following limits of $f(t)$:

(i) $\lim_{t\to\infty} f(t) =$ **68**

(ii) $\lim_{t\to-\infty} f(t) =$ ∞

b. [3 points] Find a formula for $t = f^{-1}(T)$.

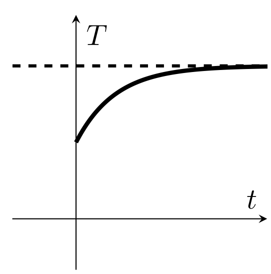
Solution: Starting with $T = 68+5e^{-0.02t}$, our goal is to isolate t, thereby finding an expression for t as a function of T.

$$\begin{aligned}
T &= 68 + 5e^{-0.02t} \\
T - 68 &= 5e^{-0.02t} \\
\frac{T-68}{5} &= e^{-0.02t} \\
\ln\left(\frac{T-68}{5}\right) &= -0.02t \\
-\frac{1}{0.02}\ln\left(\frac{T-68}{5}\right) &= -50\ln\left(\frac{T-68}{5}\right) = t
\end{aligned}$$

$f^{-1}(T) =$ $-50\ln\left(\frac{T-68}{5}\right)$

c. [3 points]

The graph of $T = q(t)$ to the right shows the temperature in a different room when being *heated* as a function of time t. The domain shown is $[0, \infty)$ and the dashed line represents a horizontal asymptote of $q(t)$.



Given that behavior, which of the following *could* be a formula for $q(t)$? (*Circle all that apply.*)

$q(t) = 3\log(t+2)$

$q(t) = 50 \cdot 1.02^t$

$\boxed{q(t) = -0.7^t + 65}$

$q(t) = -3\log(t+5)$

$\boxed{q(t) = -e^{-0.2t} + 67}$

$q(t) = -e^{0.1t} + 69$

Solution: As this graph has a horizontal asymptote, it cannot be either of the functions involving log, which only have vertical asymptotes. Further, it cannot be $q(t) = 50 \cdot 1.02^t$, because this is simply an exponentially increasing function and should only have a horizontal asymptote at $T = 0$.
Of the remaining options, $q(t) = -0.7^t + 65$ and $q(t) = -e^{-0.2t} + 67$ are both exponentially decreasing functions that have been reflected over the t-axis and then shifted up: exactly the behavior we see in the desired graph. The last option $q(t) = -e^{0.1t} + 69$ is an exponentially *increasing* function that has reflected over the t-axis and then shifted up, which is not what we see in our desired graph.

6. [10 points] For parts (a)–(d), indicate if each of the following statements is true, false, or if there is not enough information, by circling the correct answer. **Provide a *brief* explanation of your answer.**

a. [2 points] If the function $f(x)$ is odd, then the function $q(x) = (f(x))^2$ is even.

$\boxed{\text{TRUE}}$ FALSE NOT ENOUGH INFORMATION

Explanation:
$q(-x) = (f(-x))^2 = (-f(x))^2 = (f(x))^2 = q(x)$

b. [2 points] The function $\log(x)$ **can't** take negative numbers as inputs, but it **can** have negative numbers as outputs.

$\boxed{\text{TRUE}}$ FALSE NOT ENOUGH INFORMATION

Explanation:
The function $\log(x)$ is the inverse function of 10^x. Since 10^x can take negative numbers as inputs, $\log(x)$ can give negative numbers as outputs. However, since 10^x can never give negative numbers as outputs, $\log(x)$ can never take negative numbers as inputs.

c. [2 points] The function $f(x) = \log(x-h)+k$, where h, k are some constants, has a vertical asymptote at $x = h$.

$\boxed{\text{TRUE}}$ FALSE NOT ENOUGH INFORMATION

Explanation:
Because $\log(x)$ has a vertical asymptote at $x = 0$, and $f(x) = \log(x-h)+k$ is the graph of $\log(x)$ shifted right h and up k, it follows that $f(x)$ has a vertical asympote at $x = h$. (Note that this is even true if h is a negative number, we just shift "right" by a negative number, so actually shift left.)

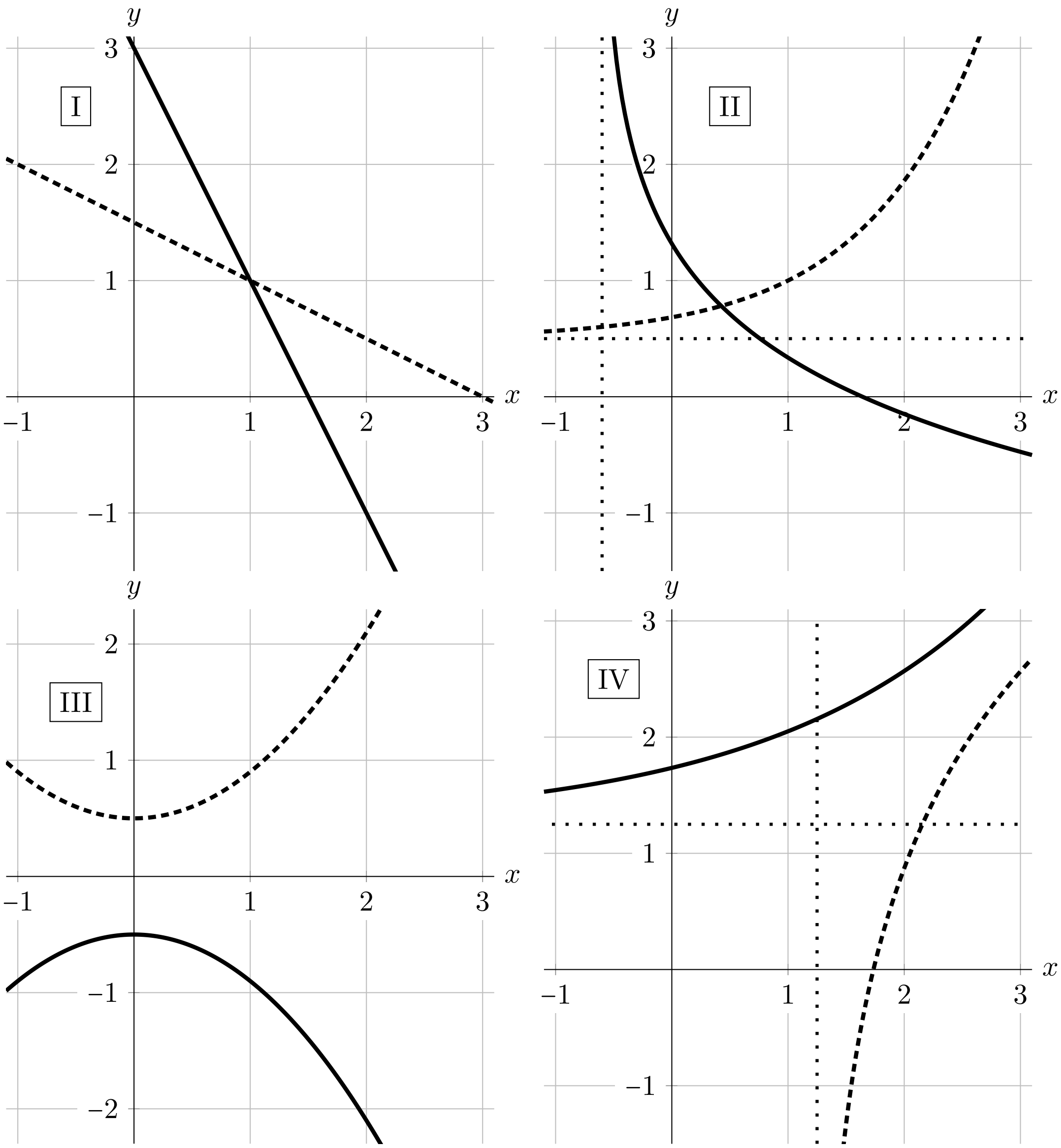
d. [2 points] If $Q(t)$ is an exponentially growing function, then the time it takes for the quantity to double gets shorter and shorter as time goes on.

TRUE $\boxed{\text{FALSE}}$ NOT ENOUGH INFORMATION

Explanation:
Any exponentially growing function has a constant doubling time, so it's doubling time does not get shorter and shorter.

This problem continues on the next page.

e. [2 points] On each set of axes below, a solid function and a dashed function are plotted. Dotted lines represent vertical or horizontal asymptotes.



For which pairs of functions shown is the solid function the inverse of the dashed function? *Circle all that apply. No justification required.*

$\boxed{\text{I}}$ II III $\boxed{\text{IV}}$