Math 105 — Final Exam — April 25, 2024

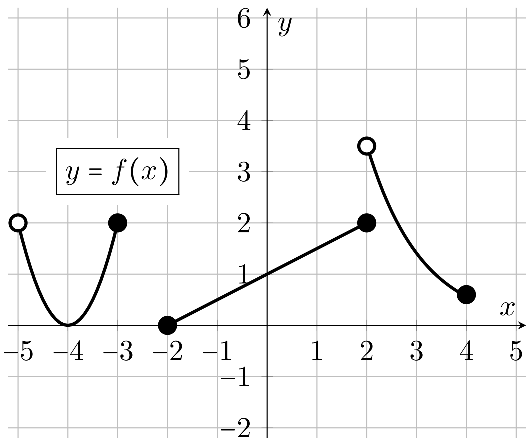
EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 13 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratch-work. Clearly identify any of this work that you would like to have graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are allowed notes written on two sides of a $3'' \times 5''$ note card and one scientific calculator that does not have graphing or internet capabilities.
10. Include units in your answer where that is appropriate.
11. Problems may ask for answers in *exact form* or in *decimal form*. Recall that $\sqrt{2} + \cos(3)$ is in exact form and 0.424 would be the same answer expressed in decimal form.
12. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

Problem	Points	Score
1	9	
2	9	
3	8	
4	11	
5	6	

Problem	Points	Score
6	8	
7	10	
8	11	
9	8	
Total	80	

1. [9 points] The entire graph of a function $f(x)$ is shown below to the left. Also shown is a table of some values for an invertible function $g(x)$, and formula for a function $h(x)$.



x	-3	-2	-0.5	0	1	2
$g(x)$	6	-5	0	4	7	9

$$h(x) = \begin{cases} \dfrac{1}{x}, & -\infty < x < 0 \\ \cos(\pi x), & 0 \le x < \infty \end{cases}$$

a. [2 points] Find the **domain** of $f(x)$. Give your answers using interval notation or using inequalities. *You do not need to explain or justify your answer.*

Domain: $(-5,-3] \cup [-2,4]$

b. [2 points] Find the **range** of $h(x)$ (the function given by a **formula**). Give your answers using interval notation or using inequalities. *Show all work, including any computations or graph sketches.*

Solution: For negative values of x, the range of $h(x)$ will be $(-\infty, 0)$. For positive values of x, the range of $h(x)$ will be $[-1,1]$. Putting those together, we get $(-\infty, 1]$.

Range: $(-\infty, 1]$

c. [5 points] Find the value of each of the following; write N/A if a value does not exist or there is not enough information to find it. *Showing work is not required, but may make you eligible for partial credit in some cases.*

(i) $g(f(2)) =$ $g(2) = 9$

(ii) $h(g^{-1}(0)) =$ $h(-0.5) = \frac{1}{-0.5} = -2$

(iii) All x such that $h(x) = -5$, $x =$ $-\frac{1}{5}$

(iv) $g(h(2)) =$ $g(\cos(2\pi)) = g(1) = 7$

(v) If $q(x) = \frac{2}{3}f(x-2)$, then $q(-1) =$ $\frac{2}{3}f(-1-2) = \frac{2}{3}f(-3) = \frac{2}{3} \times 2 = \frac{4}{3}$

2. [9 points] The parts of this problem are unrelated.

a. [6 points] Consider the quadratic function $f(x) = (x-1)^2 - 1$.

(i) Find the zero(s) and vertex of $f(x)$. *Show any relevant work.*

Solution: $f(x)$ is already given to us in vertex form, so we know that the vertex of $f(x)$ is $(1, -1)$. To find the zeros, we have a few options. One path is that we can set up the following equation and solve:

$$\begin{aligned} 0 &= (x-1)^2 - 1 \\ 1 &= (x-1)^2 \\ \pm 1 &= x - 1 \\ 1 \pm 1 = 0, 2 &= x \end{aligned}$$

vertex: $(1, -1)$ zero(s): $x =$ $0, 2$

(ii) Find the vertex of $f(2x) + 1$.

Solution: Because we we are given a transformation of f, we can identify what transformation took place, then apply those to the original vertex. The transformations that took place are a horizontal compression by $\frac{1}{2}$ and a vertical shift by 1. This means that our new vertex is at $(\frac{1}{2}, 0)$.

vertex: $(0.5, 0)$

(iii) Find the zero(s) of $3f(x-1)$.

Solution: Again, we need to consider what transformation have been applied, graphically, to the original function f. In this case, there has been a vertical stretch by a factor of 3, and a horizontal shift right by 1. The vertical stretch will not change the value of the zeros, since they will remain zero under a vertical stretch. But the shift right will do just that, shift them right. So our new zeros will be $x = 1, 3$

zero(s): $x =$ $1, 3$

b. [3 points] A different quadratic function, $g(x)$, has its vertex at $(2, 3)$ and a zero at $x = -1$.

(i) What is the x-coordinate of the other zero of $g(x)$? (CIRCLE ONE)

0 1 2 3 4 $\boxed{5}$ NONE OF THESE NOT ENOUGH INFORMATION

Solution: Any parabola has a vertical axis of symmetry. In this case, this axis of symmetry is given by $x = 2$, which we know from the given vertex. Any zeros are equidistance from the axis of symmetry. One zero is at -1, which is a distance of 3 away from $x = 2$. Thus the second zero is a distance 3 away from $x = 2$ in the other direction. In otehr words, it's $x = 5$.

(ii) What is the sign of the leading coefficient of $g(x)$? (CIRCLE ONE AND EXPLAIN)

$\boxed{\text{negative}}$ positive zero NOT ENOUGH INFORMATION

Explanation of (ii):

Solution: Because the vertex is *above* the zeros, the parabola must be "downward facing" or concave down. This means that its leading coefficient must be negative.

3. [8 points] A local clothing store *Amaizing T-Shirts* sets the price per t-shirt based on how many t-shirts a customer purchases. Let $P(s)$ be the **price per t-shirt**, in dollars, when a customer purchases s t-shirts. Note that $P(s)$ is an invertible function.

a. [2 points] Describe the meaning of $P^{-1}(10)$ in the context of the problem.

Solution: $P^{-1}(10)$ is the number of shirts that must be purchased in order for each shirt to cost \$10.

b. [2 points] Write expression for the **total price** for 156 t-shirts. Your answer may involve P and/or P^{-1}.

Answer: \$ $156 * P(156)$

c. [4 points] *Amaizing T-Shirts* offers a customer loyalty program. When a customer buys s t-shirts, they get $L(s)$ loyalty points. Describe the meaning of the following equations or explain why they don't make sense in context:

$L(100) = 10$

Solution: When a customer buys 100 shirts, they get 10 loyalty points.

$P(L^{-1}(20)) = 9$

Solution: When a customer gets 20 loyalty points, it means they've purchased enough shirts for them to cost \$9 each.

4. [11 points] As stated in the problem above, the price of each t-shirt $P(s)$ (in dollars) at *Amaizing T-Shirts* is a function of s, the total number of t-shirts a customer orders. In particular, now assume that if the customer orders exactly 1 t-shirt, it costs \$14.50. If the customer orders 30 t-shirts, each shirt costs \$13.30.

a. [3 points] If we assume that $P(s)$ is a **linear function**, find a formula for $P(s)$ *Show all work. Numbers in your final function can be rounded to two decimal places or expressed in exact form.*

Solution: We are essentially looking for a linear equation that goes through the points $(1, 14.5)$ and $(30, 13.30)$. First, we can compute the slope:

$$m = \frac{13.30 - 14.50}{30 - 1} = \frac{-1.20}{29} \approx -0.04138$$

Then, one of the simplest ways to complete our linear equation is to use point-slope form, based off of either given point. One such solution is shown below.

$$P(s) = \underline{\frac{-1.2}{29}(s-1) + 14.5 \approx -0.041s + 14.54}$$

b. [2 points] What is the meaning of the slope of $P(s)$ in your linear function above?

Meaning of Slope:

Solution: The slope found above represents the change in price-per-shirt for each additional shirt added to the total. That is, for each additional shirt added to the total purchase, the price-per-shirt will decrease by a little less than 4 cents.

c. [3 points] If we assume that $P(s)$ is an **exponential function**, find a formula for $P(s)$ *Show all work. Numbers in your final function can be rounded to two decimal places or expressed in exact form.*

Solution: We are essentially looking for an exponential equation that goes through the points $(1, 14.5)$ and $(30, 13.30)$. That is, we're looking for a function of the form ab^s. We can use our two known points to come up with two different equations, and then solve for the unknown parameters a, b.

$$13.30 = a \cdot b^{30} \qquad 14.50 = a \cdot b^1 = ab$$

If we divide the left equation by the right we get:

$$\frac{13.3}{14.5} = \frac{ab^{30}}{ab} = b^{29}$$

Thus $b = \left(\frac{13.3}{14.5}\right)^{\frac{1}{29}} \approx 0.997$

We can use that value for b to then solve for a:

$$14.50 = a \cdot \left(\frac{13.3}{14.5}\right)^{\frac{1}{29}}$$

so $a = 14.5 \cdot \left(\frac{13.3}{14.5}\right)^{-\frac{1}{29}} \approx 14.543$

$$P(s) = \underline{14.5\left(\frac{13.3}{14.5}\right)^{-\frac{1}{29}} \cdot \left(\frac{13.3}{14.5}\right)^{\frac{s}{29}} \approx 14.543 \cdot 0.997^s}$$

d. [3 points] If we assume that $P(s)$ is a **power function**, find a formula for $P(s)$ *Show all work. Numbers in your final function can be rounded to two decimal places or expressed in exact form.*

Solution: We are essentially looking for a power function equation that goes through the points $(1, 14.5)$ and $(30, 13.30)$. That is, we need to find the parameters k and p so that $y = ks^p$ will go through the given points. We can use our two known points to come up with two different equations, and then solve them for the unknown parameters.

$$13.30 = k \cdot 30^p \qquad\qquad 14.50 = k \cdot 1^p = k$$

Therefore we know right away that $k = 14.5$. We can use that then to solve for p in the first of the two equations:

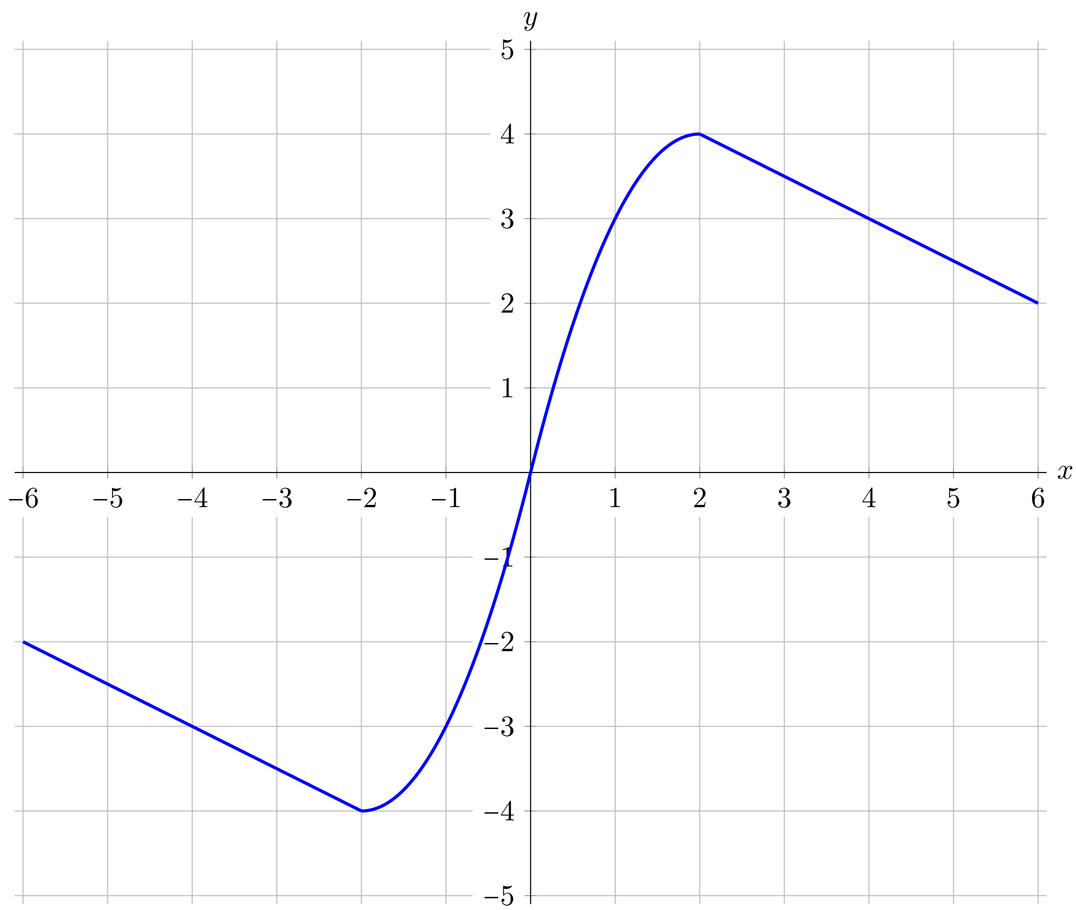
$$13.30 = 14.5 \cdot 30^p$$
$$\frac{13.3}{14.5} = 30^p$$
$$\log\left(\frac{13.3}{14.5}\right) = p\log(30)$$
$$\frac{1}{\log(30)} \cdot \log\left(\frac{13.3}{14.5}\right) \approx -0.025 = p$$

A negative value for p makes sense, because our function is decreasing. Our final formula can be seen below:

$P(s) =$ $14.5s^{\frac{1}{\log(30)} \cdot \log\left(\frac{13.3}{14.5}\right)} \approx 14.5s^{-0.025}$

5. [6 points] On the axes below, sketch the graph of a **single** function $y = f(x)$ with all of the following properties:

- $f(2) = 4$
- $f(x)$ has domain $[-6, 6]$.
- $f(x)$ is an odd function.
- $f(x)$ is concave down **and** increasing on the domain $(0, 2)$.
- $f(x)$ is linear with constant slope $-\frac{1}{2}$ on domain $[2, 6]$.



Solution: There are many different possible graphs that satisfy the criteria. One possibility is shown above.

6. [8 points] For each of the statements below, circle TRUE or FALSE and *briefly explain* your reasoning. *Credit will only be given for a reasonable explanation—circling alone is no credit.*

a. [2 points] A degree-5 polynomial will always have 5 zeros.

TRUE $\boxed{\text{FALSE}}$ NOT ENOUGH INFORMATION

Explanation:
One degree 5 polynomial with only one zero is $y = x^5$. However, that single zero *does* have a multiplicity of 5. But another example is $y = x^5 + 1$, and that has only 1 linear zero: $x = -1$. So not all degree 5 polynomials have 5 zeros, not even when accounting for multiplicity.

b. [2 points] Because a right triangle cannot contain any angles greater than 90°, we cannot find the sine or cosine of any angles greater than 90°.

TRUE $\boxed{\text{FALSE}}$ NOT ENOUGH INFORMATION

Explanation:
When we define cos and sin with the unit circle, we can find the sine and cosine of any angle, not just those between 0° and 90°

c. [2 points] If $f(x)$ is an even function with domain $[-4, 4]$, then $f(x)$ is *definitely not* invertible.

$\boxed{\text{TRUE}}$ FALSE NOT ENOUGH INFORMATION

Explanation:
Because $f(x)$ is even, then $f(-4) = f(4)$. Thus the same output is acheived twice, the function does not pass the horizontal line test, and cannot be invertible.

d. [2 points] The function $w(r) = 0.4e^{0.15r}$ is an exponentially decreasing function of r.

TRUE $\boxed{\text{FALSE}}$ NOT ENOUGH INFORMATION

Explanation:
The function $w(r)$ is actually an exponentially *increasing* function of r. This is because when the continuous growth rate $k > 0$, we have exponential growth. Another way to see this is that $b = e^0.15 > 1$, so it must be increasing/growing.

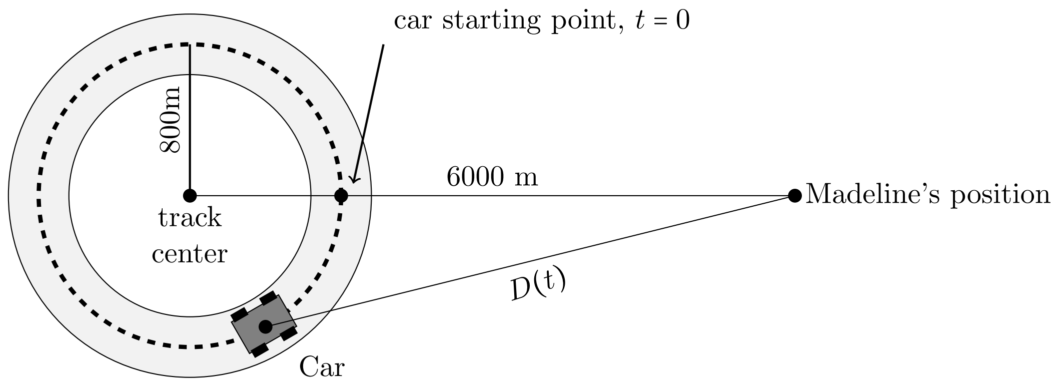
7. [10 points] Madeline is the head engineer of a new racetrack called the "Michigan Raceway". Unusually, this track will be a perfect circle. The radius a car's path will be 800m.

a. [2 points] While driving the test car, Madeline drives at a perfectly constant speed and makes one complete revolution in 1.5 minutes. How far, in meters, does Madeline drive in 40 seconds? *Show all work. Give your answer in exact form, or rounded to two decimal places.*

Solution: In 40 seconds she would have driven $40/90 = 4/9$ of a rotation. Since a full circumference is 1600π meters, she would have driven $\frac{4}{9} \cdot 1600\pi \approx 2234.021$ meters in 40 seconds.

$\frac{4}{9} \cdot 1600\pi \approx 2234.021$ meters

b. [4 points] While another engineer takes over the test car, Madeline stands directly east, 6000m away from **center** of the track. The car now drives at a constant speed and takes exactly 2 minutes for each lap. Let $D(t)$ be the test car's distance from Madeline, in meters, where t is measured in minutes since the car started, directly east of center.



(i) What is the minimum value of $D(t)$? 5200 meters

(ii) What is the maximum value of $D(t)$? 6800 meters

(iii) What is the exact value of $D(0.5)$? *Show all work.*

Solution: At 0.5 minutes, the car is exactly one quarter into a rotation, so either at the far north point of the track shown, or the far south point (depending on which way the car is going). In either case, the car's distance from Madeline can be found as the length of the hypotenuse of a right triangle. One of its side lengths is 800m, the other is 6000m. So the hypotenuse length (and hence $D(0.5)$) is

$$\sqrt{800^2 + 6000^2} \approx 6053.10 \text{ meters}$$

$\sqrt{800^2 + 6000^2} \approx 6053.10$ meters

c. [4 points] If we want to *approximate* a formula for $D(t)$ using a sinusoidal function of the form $D(t) = A\cos(B(t-h)) + k$, what formula should we use? That is, use the information above to find the appropriate values for A, B, h, and k and write out the final formula.

$D(t) = -800\cos(\pi t) + 6000$

8. [11 points]

a. [6 points] Find the following limits. They will either be a real number, ∞, or $-\infty$.
You don't need to show work but partial credit may be awarded for work shown.

(i)

$$\lim_{x\to-\infty} \frac{x^3+x^2+100}{3x^2-x} = \underline{\quad -\infty \quad}$$

(ii)

$$\lim_{x\to\infty} \frac{(7x-2)^2(x+3)}{5x^3+1} = \underline{\quad 49/5 \quad}$$

(iii)

$$\lim_{x\to\infty} \frac{e^x - x^4}{-(10^x)+x^2} = \underline{\quad 0 \quad}$$

b. [5 points] Consider the rational function:

$$f(x) = \frac{3x^2(x-3)}{(x+4)(x-1)^2(x-3)}$$

Find the following features or write NONE is none exist. *Show all relevant work.*

(i) Coordinates of any hole(s): $\underline{\quad (3, 27/28) \quad}$

(ii) Equations for any horizontal asymptote(s): $\underline{\quad y = 0 \quad}$

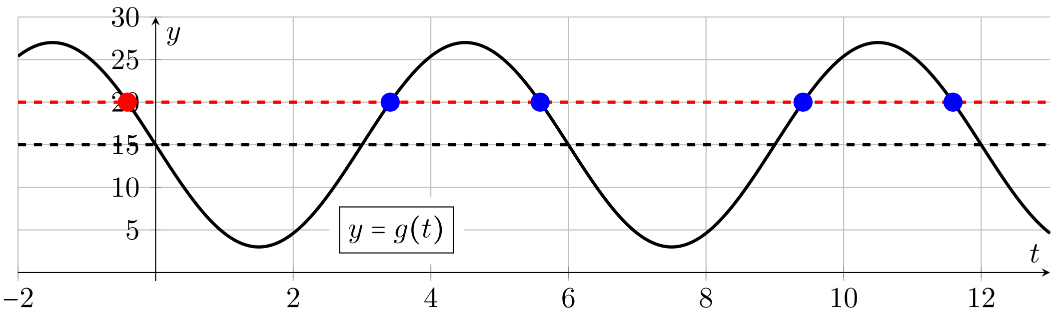
(iii) Equations for any vertical asymptotes(s): $\underline{\quad x = -4,\ x = 1 \quad}$

9. [8 points] The following problem parts are not related.

a. [3 points] The function $g(t)$, shown in the graph below, is a sinusoidal function with

- period 6
- midline $y = 15$
- and y-intercept $(0, 15)$

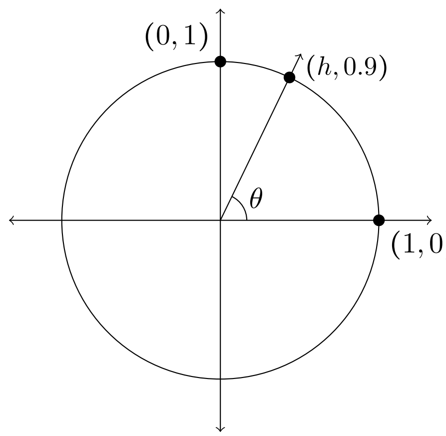
Using the fact that $g(-0.41) = 20$, find all other solutions to $g(t) = 20$ on the domain $[0, 12]$ and illustrate on the graph where they fall using dots.



Solution: We added the given solution with a red dot in the graph above. We know that two of the needed solutions will be one period, and two periods, respectively, further in the positive direction. Those will be $-0.41+6 = 5.59$ and $-0.41+12 = 11.59$. These are are the second and fourth dots from the left in the diagram above. To find the t-coordinates of the remaining solutions (the 1st and 3rd blue dots from the left), we need to use graph symmetry. There are several ways to approach that. One is to notice that the given solution, in red, is as far *before* $t = 0$ as the next solution is *after* $t = 3$ (the next time the graph crosses the midline). This makes that solution 3.41. And then a last solution is one period later: $3.41 + 6 = 9.41$.

$t =$ **3.41, 5.59, 9.41, 11.59**

b. [5 points] In the unit circle shown below, the ray at angle θ to the positive x-axis intersects the unit circle at the coordinates $(h, 0.9)$.



(i) What is the value of h? *Show all relevant work. Give your final answer in exact form, or accurate to two decimal places.*

Solution: $h^2 + (0.9)^2 = 1 \implies h = \sqrt{1-(0.9)^2} \approx 0.44$

$h =$ $\sqrt{1-(0.9)^2} \approx 0.44$

(ii) What is the value of θ in degrees? *Show all relevant work. Give your final answer in* ***numerical*** *form, accurate to two decimal places.*

Solution: $\sin(\theta) = 0.9 \implies \theta = \arcsin(0.9) = 64.16°$, or $180° - 64.16°$. Since the angle is in the first quadrant, the correct answer is $64.16°$.

$\theta =$ 64.16 $^\circ$

(iii) Find the value for an angle ϕ (in degrees), between $90°$ and $180°$, such that $\sin(\phi) = 0.9$. *Show all relevant work. Give your final answer in terms of θ or as a number rounded to two decimal places.*

Solution: $\sin(\theta) = 0.9 \implies \theta = \arcsin(0.9) = 64.16°$, or $180° - 64.16°$. Since the angle is in the second quadrant, the correct answer is $180° - 64.16°$.

$\phi =$ 115.84 $^\circ$

(iv) Find **all** possible values of an angle ω (in degrees) between 0 and $360°$ such that $\cos(\omega) = 0.9$. *Show all relevant work. Give your final answer in terms of θ or as a number rounded to two decimal places.*

Solution: **Solution 1.** $\cos(\omega) = 0.9 \implies \omega = \arccos(0.9) = 25.84° + 360°k$, or $360°k - 25.84°$. The only angles among these that are in the interval $0°$ to $360°$ are $25.84°$ and $360° - 25.84°$.
Solution 2. At the angle θ, $\sin(\theta) = 0.9$ so $\cos(90° - \theta) = 0.9$. If $\cos(\omega) = 0.9$, then $\cos(-\omega) = 0.9$ also, since cos is even. So $\theta - 90°$ is also a solution, but it's not in the interval $0°$ to $360°$, so we add $360°$ to get $\theta + 270°$.

$\omega =$ $25.84°$**,** $360° - 25.84°$ **or** $90 - \theta$**,** $\theta + 270$ $^\circ$