Math 105 — First Midterm — February 17, 2025

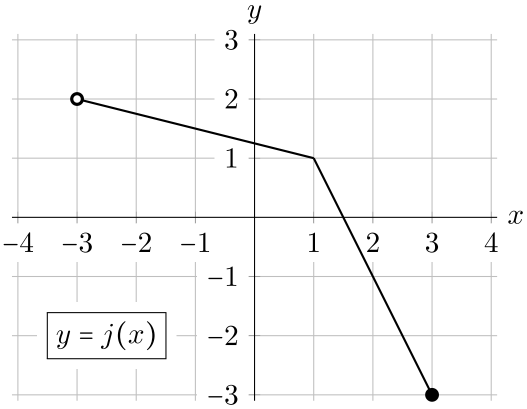
EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 9 pages including this cover. There are 6 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratch-work. Clearly identify any of this work that you would like to have graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are allowed notes written on two sides of a $3'' \times 5''$ notecard and one scientific calculator that does not have graphing or internet capabilities.
10. Include units in your answer where that is appropriate.
11. Problems may ask for answers in *exact form* or in *decimal form*. Recall that $\sqrt{2} + \cos(3)$ is in exact form and 0.424 would be the same answer expressed in decimal form.
12. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is <u>not</u> permitted.

Problem	Points	Score
1	9	
2	8	
3	11	

Problem	Points	Score
4	11	
5	10	
6	11	
Total	60	

1. [9 points] The entire graph of a function $j(x)$, which is made up of two linear pieces, is shown below to the left. Also shown is a table of some values for a different function $k(x)$. Assume that the function $k(x)$ is invertible.



x	−3	−1	0	1	3	4
$k(x)$	−5	−3	−1	0	4	7

a. [2 points] Find the domain and range of $j(x)$. Give your answers using either interval notation or using inequalities. *You do not need to explain or justify your answer.*

Answer: $j(x)$ has domain $(-3, 3]$ and range $[-3, 2)$

b. [7 points] Find the **exact** value of each of the following, or write N/A if a value does not exist or there is not enough information to find it exactly. *You do not need to show work.*

i. $k^{-1}(4) =$ 3

ii. $j^{-1}(3) =$ **N/A**

iii. $j(k(1)) = \frac{5}{4}$ **or** 1.25

iv. $m(0)$, where $m(x) = k(x-1) + 3$

Answer: $m(0) =$ 0

v. all values of x so that $k(j(x)) = -3$

Answer: $x =$ 2

vi. the average rate of change of $k(x)$ on the interval $[-3, 4]$

Answer: $\frac{12}{7}$

2. [8 points] In the year 2000, the population of the town of Ashford was 3 thousand people. By 2009, it had grown to 7 thousand.

For parts **a**. and **b**. below, show any needed work and give numerical values exactly or rounded to at least three decimal places.

a. [2 points] Assuming that the population of Ashford, in thousands, is a linear function $L(t)$ of t, the number of years since 2000, write a formula for $L(t)$.

$L(t) =$ $3 + \frac{4}{9}t$ **or** $3 + 0.444t$

b. [3 points] Now, instead, assuming that the population of Ashford, in thousands, is an exponential function $E(t)$ of t, the number of years since 2000, write a formula for $E(t)$.

$E(t) =$ $3\left(\frac{7}{3}\right)^{t/9}$ **or** $3(0.920)^t$

The population of a neighboring town, Beaumont, has increased by 3% each year since 2000.

c. [3 points] Completely fill in the circle corresponding to the **one** best description of the function $P(t)$ that gives the population of Beaumont, in thousands, as a function of t, the number of years since 2000.

○ $P(t)$ is linear because it has a slope of 0.03.

○ $P(t)$ is linear because the population increases at a constant 3% rate.

○ $P(t)$ is exponential because it increases by 30 people each year on average.

● $P(t)$ is exponential because it has a constant growth factor of 1.03.

○ There is not enough information to determine whether $P(t)$ is linear or exponential.

3. [11 points] The cost and amount of memory in computers has changed dramatically over time. For the t^{th} year after 2000,

- let $C(t)$ be the cost, in dollars, of 1 gigabyte (GB) of memory in year t, and
- let $M(t)$ be the average amount of memory, in GB, in a new computer in year t.

a. [2 points] The function $C(t)$ is decreasing over its domain $t \geq 0$. Briefly explain why this means that C must be invertible, that is, why the inverse C^{-1} must also be a function.

Solution: Since C is decreasing, as we move to the right the output values get smaller, so no two output values come from the same input. For C^{-1}, then, no input will have more than one associated output, as needed.

Alternately, since C is decreasing, as time went on the cost of memory decreased, so no year had the same cost. This means we can also view the year as a function of the cost, and this is the function C^{-1}.

b. [2 points] Suppose that the average rate of change of $C(t)$ over the interval $[10, 16]$ was -1.3. Interpret what this means in the context of the problem.

Solution: Between the years 2010 and 2016, the cost of 1 GB of memory decreased by $1.30 each year on average.

c. [5 points] Assume that M and C are both invertible functions. Describe the meaning of each of the following expressions or equations in the context of this problem, or explain why the expression or equation doesn't make sense in context.

i. $M(15) = 8$

Solution: In 2015, the average amount of memory in a new computer was 8 GB.

ii. $M^{-1}(4)$

Solution: the number of years after 2000 when the average amount of memory in a new computer was 4 GB

iii. $M(C(10))$

Solution: This does not make sense in context, because C outputs a cost in dollars, while M expects a number of years as its input.

d. [2 points] Write down a mathematical equation that represents the following sentence.

When the average amount of memory in a new computer was $\frac{1}{2}$ GB, the average cost of each GB was $100.

Answer: $C\left(M^{-1}\left(\frac{1}{2}\right)\right) = 100$

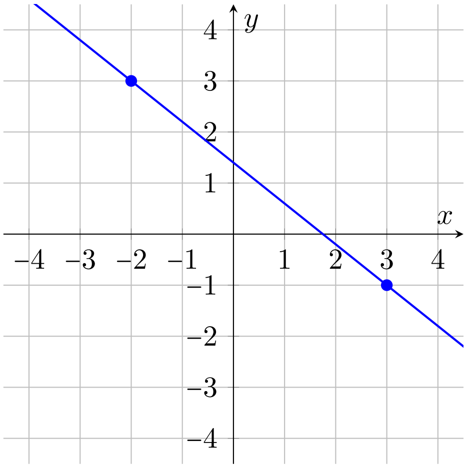
Note: there are other equivalent answers.

4. [11 points] For each part below, carefully draw on the axes to the right the graph of a single function that satisfies the given conditions, or, if it is not possible to do so, write NOT POSSIBLE and briefly explain why.

Make sure that any graph you draw is clear and unambiguous, and that you have carefully plotted any important points.

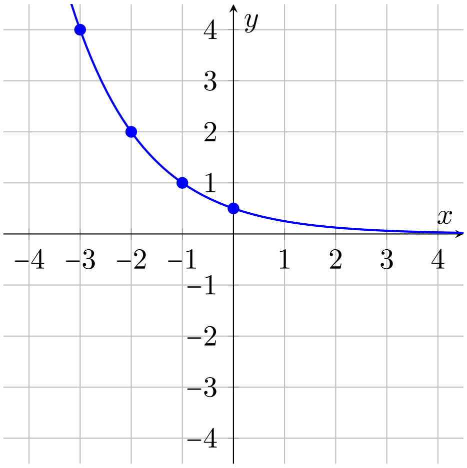
a. [2 points]

A linear function that passes through the point $(3, -1)$ and has a slope of $-\frac{4}{5}$.



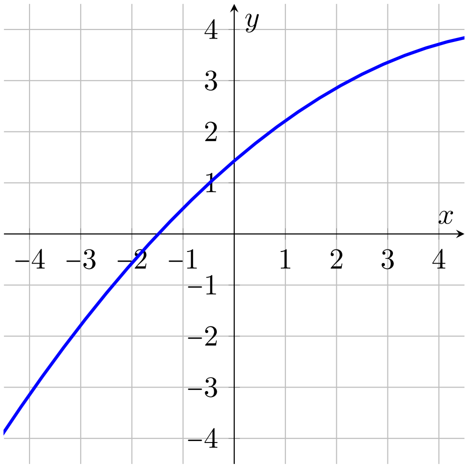
b. [3 points]

An exponential function that passes through the point $(-2, 2)$ and has a decay rate of 50%.



c. [2 points]

A function that is concave down and increasing for $-4 < x < 4$.



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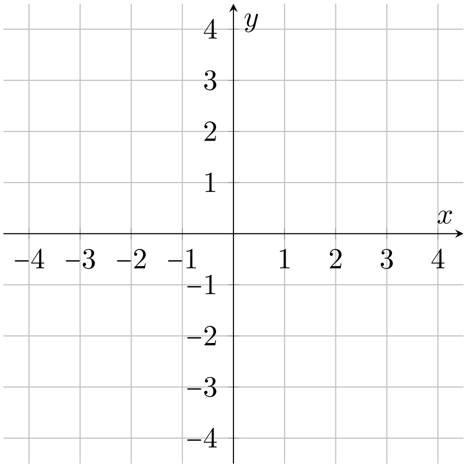
For each part below, carefully draw on the axes to the right the graph of a single function that satisfies the given conditions. Make sure your graph is clear and unambiguous. Or, if it is not possible to do so, write NOT POSSIBLE and briefly explain why.

d. [2 points]

A function that is concave up for $-4 < x < 4$ and that has the values given in the following table:

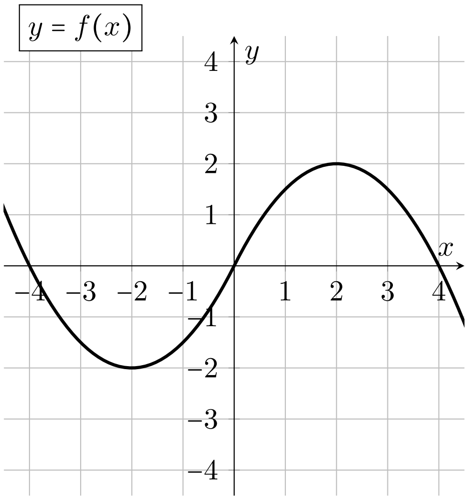
x	-2	0	3
y	-2	1	4

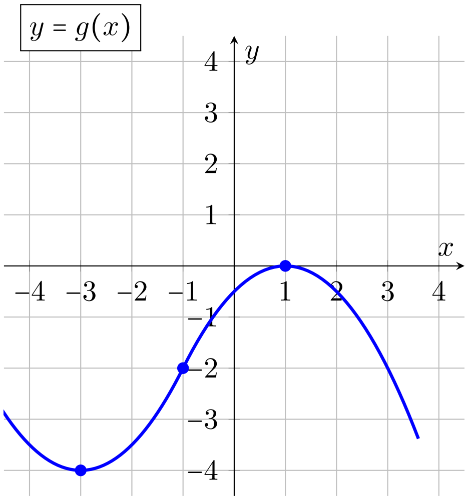
Solution: NOT POSSIBLE. The average rate of change on $[0, 3]$ is smaller than that on $[-2, 0]$, so the function cannot be concave up.



e. [2 points]

The function $g(x) = f(x+1) - 2$, where $f(x)$ is given in the graph below to the left.





5. [10 points] Leena climbs a tree, picks an apple, and throws it into a basket that is on the ground 9 meters away from the base of the tree. The apple's path is the shape of a parabola, reaching its peak of 10 meters above the ground when its horizontal location is 4 meters away from the tree. Let $H(m)$ be the function that gives the apple's height, in meters, when its horizontal location is m meters away from the tree.

a. [3 points] Write a formula for $H(m)$ in vertex form. Show any needed work.

Solution: The vertex is $(4, 10)$, so we know $H(m) = a(m-4)^2 + 10$. Then, since we must have $H(9) = 0$, we can solve $0 = a(9-4)^2 + 10$ to find that $a = -\frac{2}{5}$.

Answer: $H(m) =$ $-\frac{2}{5}(m-4)^2 + 10$

b. [3 points] Write a formula for $H(m)$ in factored form. Show any needed work.

Solution: Since 9 is one zero and the vertex is at $m = 4$, the other zero of the parabola must be -1 by symmetry. We can solve for a similarly to part **a**. or just note that the a-value must be the same.

Answer: $H(m) =$ $-\frac{2}{5}(m-9)(m+1)$

c. [2 points] Find the value of $H(0)$ in decimal form, then interpret what it means in the context of this problem.

Answer: $H(0) =$ 3.6

Interpretation:

Solution: Leena was 3.6 meters above the ground when she threw the apple.

d. [2 points] Leena attempts to throw an apple to her friend Toya, who is behind a wall that is 10 meters away and 2 meters tall. Her throw sets the apple on a path such that, when its horizontal location is m meters away from her, its height in meters is given by $L(m) = -0.2m^2 + 1.5m + 6$. Does the apple get to Toya? Circle the correct answer and provide a brief explanation.

Yes [✓ **No**]

Explanation:

Solution: We can find that $L(10) = 1$, so the height of the apple is only 1 meter when it reaches the fence. It will hit the fence rather than go over it, since the fence is 2 meters tall. Or, one could also find the time at which the apple is 2 meters off the ground by solving $L(m) = 2$. Since the positive solution is about 7.8 meters, we know that the apple will be at that height well before it reaches the fence horizontally.

6. [11 points] Jose's business is now selling fuzzy gloves. He has 100 pairs to sell for \$6 a pair. Jose's friend Neil has his own business making fuzzy fabric and makes a deal with Jose: if Jose sells all 100 pairs of gloves, Neil will provide fabric that will allow Jose to sell up to 300 additional pairs for \$9 a pair, as long as Neil gets \$2 from each pair sold.

Jose wants a function $J(p)$ for the amount of money, in dollars, he would make if he sold p pairs of gloves and gave Neil his share, if applicable.

a. [1 point] What is the domain of J in the context of this problem? Use either inequality or interval notation.

Domain: $[0, 400]$

b. [4 points] Write a piecewise-defined formula for the function $J(p)$ on its domain.

Answer: $$J(p) = \begin{cases} 6p & \text{for } 0 \le p \le 100 \\ 600 + 7(p-100) & \text{for } 100 \le p \le 400 \end{cases}$$

Neil also wants a function $N(p)$ for the amount of money, in dollars, he would make if Jose sells p pairs of gloves.

c. [3 points] Write a piecewise-defined formula for the function $N(p)$.

Answer: $$N(p) = \begin{cases} 0 & \text{for } 0 \le p \le 100 \\ 2(p-100) & \text{for } 100 \le p \le 400 \end{cases}$$

d. [3 points] If the fabric Neil plans to give Jose costs \$500, how many pairs of gloves, in total, does Jose have to sell for Neil to recoup his costs, that is, for Neil to make \$500? Show all of your work.

Solution: We set $2(p-100) = 500$ and solve to find that $p = 350$.

Answer: 350