Math 105 — Second Midterm — March 31, 2025

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 9 pages including this cover. There are 7 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratch-work. Clearly identify any of this work that you would like to have graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are allowed notes written on two sides of a $3'' \times 5''$ notecard and one scientific calculator that does not have graphing or internet capabilities.
10. Include units in your answer where that is appropriate.
11. Problems may ask for answers in *exact form* or in *decimal form*. Recall that $\sqrt{2} + \cos(3)$ is in exact form and 0.424 would be the same answer expressed in decimal form.
12. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

Problem	Points	Score
1	10	
2	6	
3	5	
4	12	

Problem	Points	Score
5	9	
6	10	
7	8	
Total	60	

1. [10 points] The function $A(x)$ has domain $(-\infty, \infty)$, is **odd**, and is periodic with period 8. Some values of $A(x)$ are given in the table below as well as a formula for the function $B(x)$.

x	-4	-2	1	3
$A(x)$	0	3	1	-2

$$B(x) = 7\log(x^2) - 4$$

a. [6 points] Find each of the following values. *Give your answers in exact form or rounded to 3 decimal places. Or, if there is not enough information to find a value, write* NEI, *or if the value does not exist, write* DNE.

i. $A(0) =$ 0

ii. $A(2) =$ -3

iii. $A(6) =$ 3

iv. $A(25) =$ 1

v. $B(A(3)) =$ $7\log(4) - 4 \approx 0.214$

vi. $A(B(1)) =$ 0

Suppose that $C(x)$ is a different periodic function with amplitude 7 and a maximum of 11. The period of $C(x)$ is also 8.

b. [2 points] Find each of the following, or if there is not enough information, write NEI.

i. the midline of $C(x)$: $y =$ 4

ii. the minimum of $C(x)$: -3

c. [2 points] Must the function $A(x) + C(x)$ also be periodic? Circle your answer, and then briefly explain.

$\boxed{\text{YES}}$ NO

Explanation:

Solution: Since $A(x + 8) = A(x)$ and $C(x + 8) = C(x)$ for any input x, the sum $A(x + 8) + C(x + 8) = A(x) + C(x)$ is also periodic. That is, since the values of $A(x)$ repeat after 8 units, and the same for $C(x)$, the sums of these values also repeat after 8 units.

2. [6 points] Scientists are studying three different populations of bacteria.

a. [4 points] Population A is doubling every 5 hours.

Show any needed work. Give answers in exact form or rounded to 3 decimal places.

i. What is the growth factor for population A?

Solution: If a is the initial amount and b the growth factor for Population A, then $ab^5 = 2a$. Then $b = 2^{1/5} \approx 1.149$.

Answer: $2^{1/5}$

ii. How long does it take population A to triple?

Solution: We solve $a\left(2^{1/5}\right)^t = 3a$ to find that $t = \frac{\ln(3)}{\ln(2^{1/5})}$.

Answer: ≈ 7.925 hours

b. [2 points]

Population B is growing exponentially at a rate of 16% per hour.
Population C is growing exponentially at a *continuous* rate of 14% per hour.

Which of these two populations is growing more quickly? Circle your answer, then briefly justify.

$\boxed{\text{POPULATION B}}$ POPULATION C

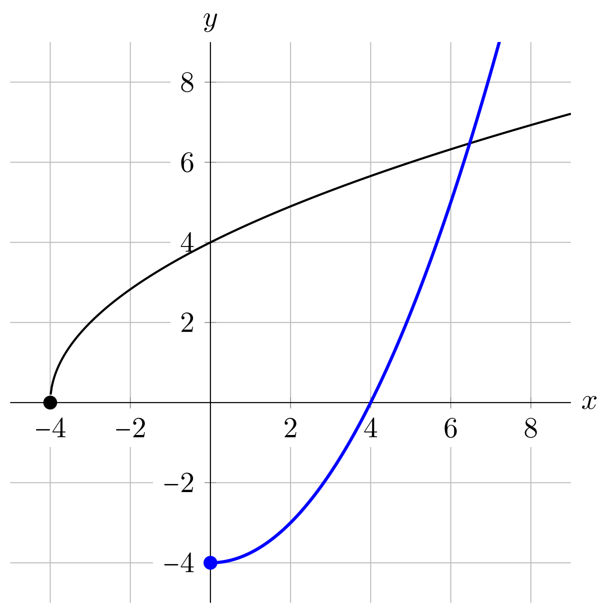
Justification:

Solution: Since $e^{0.14} \approx 1.15$, Population C is growing at a rate of about 15% per hour, compared to Population B which is growing at 16% per hour.

3. [5 points] Consider the function

$$g(x) = 2\sqrt{x+4}.$$

a. [2 points] A graph of $g(x)$ is shown below. On the same set of axes, sketch a graph of $g^{-1}(x)$. Be sure your graph is clear and unambiguous, and that you have carefully plotted any important points.



b. [3 points] Find a formula for $x = g^{-1}(y)$. Show your work.

Solution: We solve $y = 2\sqrt{x+4}$ for x to find

$$\frac{y}{2} = \sqrt{x+4}$$

$$\frac{y^2}{4} = x + 4$$

$$\frac{y^2}{4} - 4 = x$$

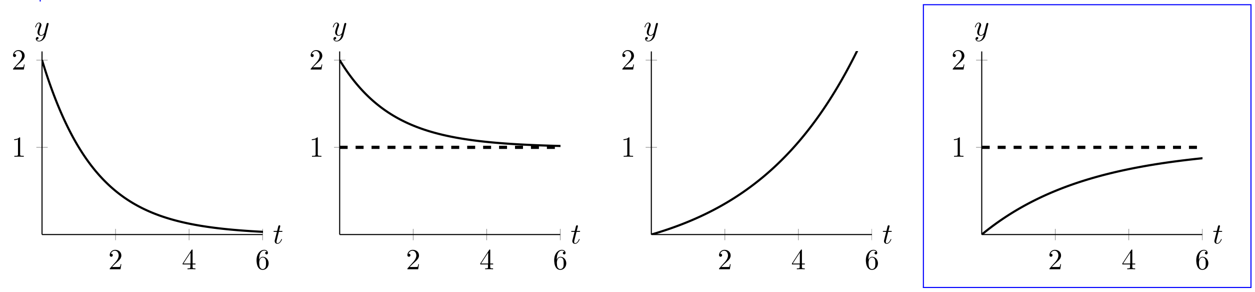
Answer: $g^{-1}(y) = \underline{\quad \frac{y^2}{4} - 4 \quad}$

4. [12 points] Nash the dog loves to eat. The amount of food, in pounds, that Nash has consumed t seconds after the start of a meal can be modeled by the function

$$N(t) = 1 - 2^{-0.5t}.$$

a. [2 points] Circle the one graph that could be the graph of $y = N(t)$.

Solution: Note that $N(0) = 0$ and $\lim_{t\to\infty} N(t) = 1$.



b. [6 points] Find a sequence of transformations you could apply to the function 2^t to transform it into $N(t)$. For each transformation, fill in the first blank with one of the options listed below. In the second blank, include the amount of any shift, or the scaling factor for any stretch or compression. For reflections, write N/A.

HORIZONTAL / VERTICAL REFLECTION
STRETCH IT HORIZONTALLY / VERTICALLY
COMPRESS IT HORIZONTALLY / VERTICALLY
SHIFT IT LEFT / RIGHT / UP /DOWN

First, **horizontal reflection** by **N/A**.

Then, **vertical reflection** by **N/A**.

Then, **horizontal stretch** by **2**.

Then, **shift up** by **1**.

Solution: Note that any order of these transformations in which the vertical reflection occurs before the shift up is also correct.

c. [4 points] Let $F(m)$ denote the number of pounds of food Nash has consumed m **minutes** after the start of a meal.

i. Find an implicit formula for $F(m)$. That is, find a formula for $F(m)$ in terms of the function N.

Answer: $F(m) =$ $N(60m)$

ii. Now find an explicit formula for $F(m)$. That is, find a formula for $F(m)$ that does *not* use the function N.

Answer: $F(m) =$ $1 - 2^{-0.5(60m)}$ **or** $1 - 2^{-30m}$

5. [9 points] Matthew bakes chocolate soufflés to sell in his restaurant, and he is testing how the soufflés cool so that he can serve them at the perfect temperature.

a. [4 points] In his home kitchen, the temperature of a soufflé, in degrees Celsius (°C), after being out of the oven for t seconds, is given by

$$H(t) = 177e^{kt} + c, \qquad \text{where } c \text{ and } k \text{ are constants.}$$

Matthew finds that the temperature of a soufflé at the moment it comes out of the oven is 195°C. After 100 seconds, it has cooled to 60°C. Find the values of c and k. *Show all your work. Give your answers in exact form or rounded to 3 decimal places.*

Solution: Since

$$H(0) = 177e^{k\cdot 0} + c = 177 + c = 195,$$

we must have that $c = 18$. Then,

$$H(100) = 177e^{k\cdot 100} + 18 = 60$$

$$e^{k\cdot 100} = \frac{42}{177}$$

$$k = \frac{1}{100}\ln\left(\frac{42}{177}\right)$$

$c =$ 18 $\qquad k =$ ≈ -0.014

b. [3 points] When he moves to his restaurant kitchen, the temperature of a soufflé, in °C, after being out of the oven for t seconds is instead given by

$$R(t) = 188e^{-0.01t} + 22.$$

After taking a soufflé out of his restaurant's oven, how long should Matthew wait to serve it if he wants it to be 80°C at that moment? *Show all your work. Give your answers in exact form or rounded to 3 decimal places.*

Solution: We set $R(t) = 80$ and solve for t:

$$188e^{-0.01t} + 22 = 80$$

$$e^{-0.01t} = \frac{58}{188}$$

$$t = \frac{\ln\left(\frac{58}{188}\right)}{-0.01}$$

117.600 seconds

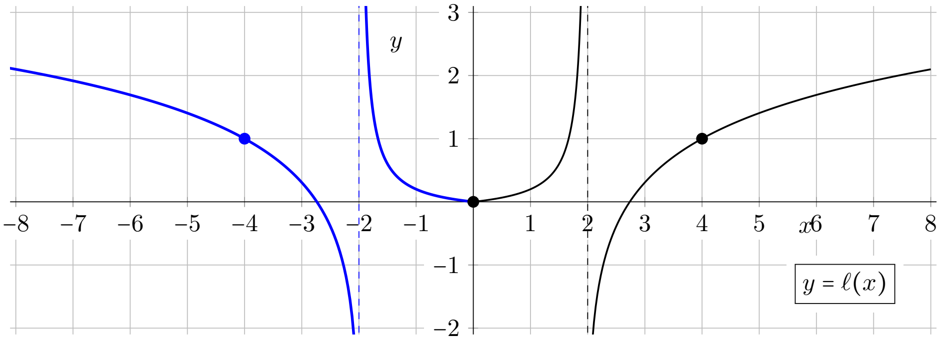
c. [2 points] Find $\lim_{t\to\infty} R(t)$, then interpret what it means in the context of this problem, including any relevant units.

$\lim_{t\to\infty} R(t) =$ 22

Interpretation:

Solution: If Matthew let the soufflé cool for a very long time, its temperature would approach 22°C.

6. [10 points] Part of the graph of an **even** function $y = \ell(x)$ is shown below. Though the graph is only drawn for the interval $[0, 8)$, the domain of ℓ is all real numbers except those where it has a vertical asymptote.



a. [2 points] On the axes above, draw in the graph of $\ell(x)$ on the interval $(-8, 0)$.

b. [2 points] On the interval $(2, \infty)$, the function $\ell(x)$ is either a transformed exponential function or a transformed logarithm function. Circle which one, then briefly justify.

EXPONENTIAL $\boxed{\text{LOGARITHM}}$

Justification:

Solution: Since this part of the graph has a vertical asymptote, it must be a transformed logarithm function, since they have vertical asymptotes and exponential functions do not.

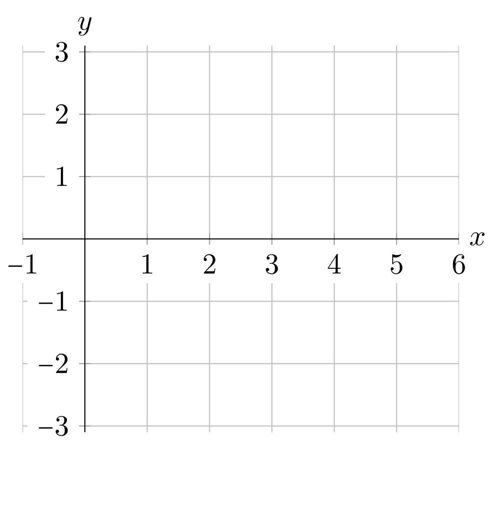
c. [2 points] Consider the function $j(x) = \ell(2x) - 7$. Is $j(x)$ even, odd, or neither, or is there not enough information to tell? Circle your answer, then briefly justify.

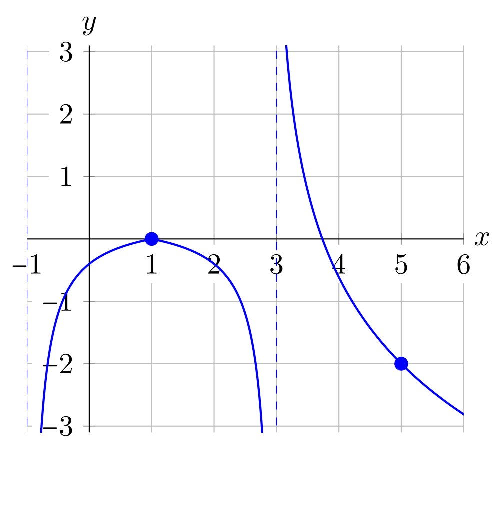
$\boxed{\text{EVEN}}$ ODD NEITHER NOT ENOUGH INFO

Justification:

Solution: We have $j(-x) = \ell(-2x) - 7 = \ell(2x) + 7 = j(x)$ since $j(x)$ is even. Or, a horizontal compression and shift up do not change the symmetry over the y-axis.

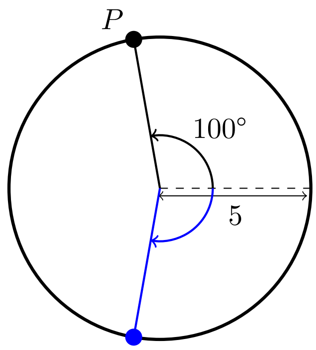
d. [4 points] Consider the function $m(x) = -2\ell((x - 1))$. Carefully sketch the graph of $m(x)$ on the entire interval $(-1, 6)$. Make sure any **asymptotes** are clear, and mark the new locations of the **closed circles** in the original graph. Give your final answer on the axes to the **right**. The extra set of axes on the left may be used for scratchwork.





7. [8 points]

a. [5 points] The point P lies on a circle with center $(0,0)$ and radius 5. P makes an angle of $100°$ with the positive horizontal axis as shown below.



i. Find the coordinates of P, either in exact form or rounded to 3 decimal places.

Answer: $P =$ $(5\cos 100°, 5\sin 100°)$

ii. Find another angle θ between $0°$ and $360°$ so that $\cos(\theta) = \cos(100°)$, then briefly justify your answer. You can draw any relevant points and/or angles on the circle above as part of your justification.

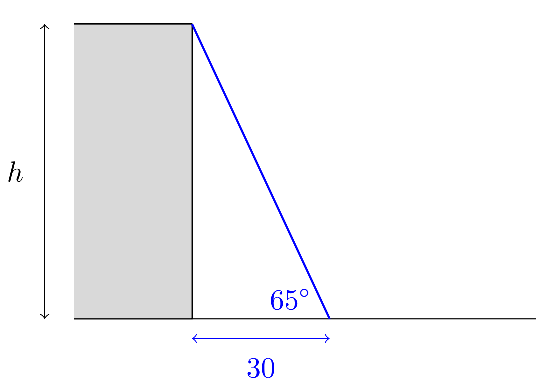
Answer: 260 °

Justification:

Solution: An angle of $360° - 100° = 260°$ will have a corresponding point on the circle with the same x-coordinate as point P above, and therefore the same value of cos. See the diagram above.

b. [3 points] You are trying to determine the height h of a building on campus, as shown below. You are 30 meters in front of the building (that is, to the right in the diagram), and from that location, you measure that the angle between the ground and the top of the building is $65°$.

Carefully draw a relevant triangle on the diagram below, then use it to find h. Be sure your triangle includes labels for any relevant distances or angles. Give your answer in exact form or rounded to 3 decimal places.



Answer: $h =$ $30\tan 65° \approx 64.335$ meters