# Math 105 — Final Exam — April 25, 2025

EXAM SOLUTIONS

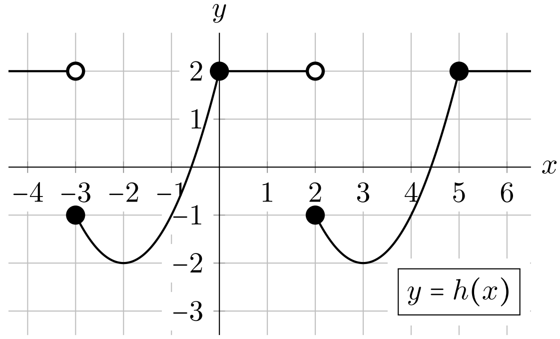
1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratch-work. Clearly identify any of this work that you would like to have graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are allowed notes written on two sides of a $3'' \times 5''$ notecard and one scientific calculator that does not have graphing or internet capabilities.
10. Include units in your answer where that is appropriate.
11. Problems may ask for answers in *exact form* or in *decimal form*. Recall that $\sqrt{2} + \cos(3)$ is in exact form and 0.424 would be the same answer expressed in decimal form.
12. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

| Problem | Points | Score |
|---|---|---|
| 1 | 11 | |
| 2 | 11 | |
| 3 | 7 | |
| 4 | 10 | |

| Problem | Points | Score |
|---|---|---|
| 5 | 7 | |
| 6 | 14 | |
| 7 | 10 | |
| 8 | 10 | |
| Total | 80 | |

**1.** [11 points] Below is a table of some values for the functions $f(x)$ and $g(x)$, and a partial graph of the function $y = h(x)$, which is periodic with period 5.

| $x$ | $-5$ | $-3$ | 1 | 3 |
|---|---|---|---|---|
| $f(x)$ | 27 | 18 | 12 | 8 |
| $g(x)$ | 12 | 15 | 21 | 24 |




For each of the following, completely fill in the circle for **all** correct answers.

**a.** [2 points] Which of these functions could be linear?

○ $f(x)$ ● $g(x)$ ○ $h(x)$ ○ NONE OF THESE

**b.** [2 points] Which of these functions could be exponential?

○ $f(x)$ ○ $g(x)$ ○ $h(x)$ ● NONE OF THESE

**c.** [2 points] Which of these functions could be invertible?

● $f(x)$ ● $g(x)$ ○ $h(x)$ ○ NONE OF THESE

**d.** [3 points] Find each of the following values exactly, or write NEI if there is not enough information to do so.

$\dfrac{f(3)}{g(3)} =$ $\frac{1}{3}$ $g\left(\frac{1}{4}f(3) - 1\right) =$ 21 $h(-23) =$ $-1$

**e.** [2 points] Find the period of the function $-h\left(\frac{1}{3}x\right)$.

**Answer:** 15

**2.** [11 points] Invasive beetles were accidentally introduced to a nature preserve, and their population then grew exponentially for 11 weeks. In particular, the number of beetles in the preserve $t$ weeks after their introduction was modeled by the function

$$b(t) = 4(1.5)^t \text{ for } 0 \le t < 11.$$

*Show your work and give answers in exact form or rounded to at least two decimal places unless otherwise noted.*

**a.** [3 points] By what percent did the beetle population grow each **day**?

*Solution:* Since $t = 1/7$ corresponds to one day, we can find $b(1/7) = 4(1.5)^{1/7}$. Dividing this number by 4 to find the percent increase from the initial value, we get $(1.5)^{1/7} \approx 1.0596$.

**Answer:** $100\left((1.5)^{1/7} - 1\right) \approx 5.96$ %

**b.** [3 points] At what time $t$ was the number of beetles equal to 100?

*Solution:* We solve $4(1.5)^t = 100$ to find that $(1.5)^t = 25$, so $t = \dfrac{\ln(25)}{\ln(1.5)}$.

**Answer:** $t = \approx 7.94$

**c.** [5 points] The beetle was detected and, after 11 weeks, eradication efforts began. From that time, the population decreased at a rate of 50 beetles per week until the population was completely removed.

i. How many beetles were there after 11 weeks?
*You may round your answer to the nearest whole beetle.*

**Answer:** $4(1.5)^{11} \approx 346$ beetles

ii. From the time the beetle population was introduced to the preserve, how many weeks passed before it was completely removed?

*Solution:* When eradication efforts began, there were 346 beetles, so it took $346/50 \approx 6.92$ weeks beyond the first 11 weeks.

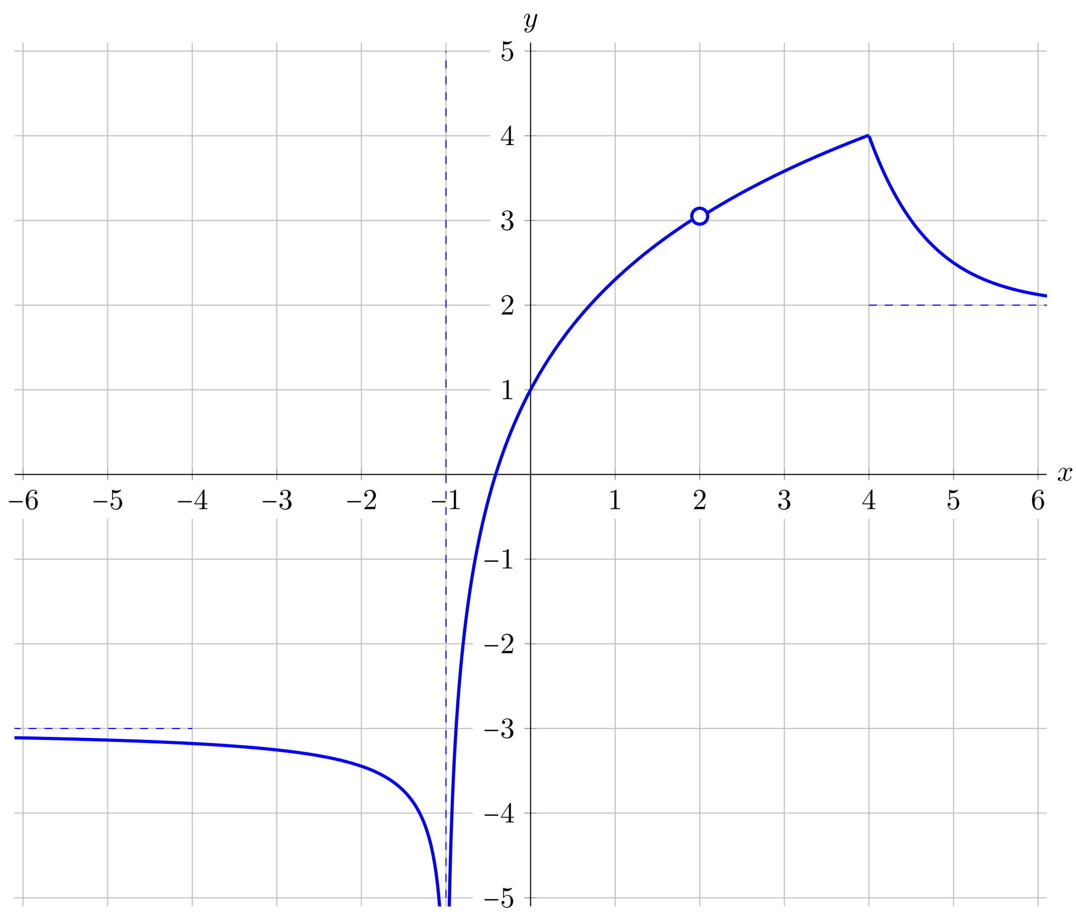
**Answer:** 17.92 weeks

iii. Use your answers to complete the piecewise formula given below for the beetle population $b(t)$ from when it was first introduced to the preserve until the time it was completely removed.

**Answer:**
$$b(t) = \begin{cases} 4(1.5)^t & 0 \le t < 11 \\ 346 - 50(t - 11) & \mathbf{11} \le t \le \mathbf{17.92} \end{cases}$$

**3.** [7 points] On the axes below, sketch the graph of a **single** function $y = f(x)$ with all of the following properties:

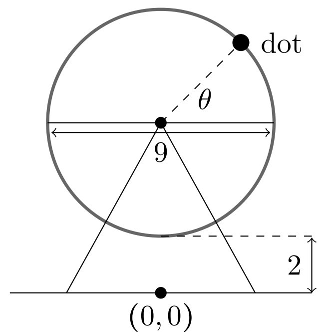
- $f(x)$ has exactly one vertical asymptote of $x = -1$
- $f(x)$ has exactly one hole at $x = 2$
- the domain of $f(x)$ is all real numbers except $-1$ and $2$
- the range of $f(x)$ is $(-\infty, 4]$
- $f(x)$ is concave up and decreasing for $4 < x < 6$
- $\lim_{x\to\infty} f(x) = 2$
- $f(x)$ has a horizontal asymptote of $y = -3$




*Solution:* There are many different possible graphs that satisfy the criteria. One possibility is shown above.

**4**. [10 points] Rupert the hamster has another new hamster wheel. It has a diameter of 9 inches and lies 2 inches off the ground. Consider the point (0,0) to be on the ground directly below the center of the wheel, as shown. Rupert's owner adds a dot to the wheel so she can track how much Rupert runs.

Just before Rupert starts running, the dot makes an angle of $\theta = \frac{\pi}{4}$ with the positive horizontal axis.




*Show your work and give all answers in exact form or rounded to at least two decimal places.*

**a**. [3 points] Find the coordinates of the dot in its current position.

**Answer:** ( $4.5\cos(\frac{\pi}{4}) \approx 3.18$ , $4.5\sin(\frac{\pi}{4})+6.5 \approx 9.68$ )

Rupert begins to run.

- The wheel turns counterclockwise, so that the dot moves from its location in the diagram up and left before continuing around the wheel.
- He runs at a constant pace so that each full turn of the wheel takes 4 seconds.
- The dot travels 3 full rotations, followed by a partial rotation so that it ends up at the **bottom** of the wheel when Rupert stops running.

**b**. [2 points] How many rotations of the wheel did the dot make?

**Answer:** $3 + \frac{5}{8} = 3.625$

**c**. [1 point] For how many seconds did Rupert run?

*Solution:* We take the number of rotations times 4 seconds per rotation.

**Answer:** 14.5 seconds

**d**. [2 points] What distance did Rupert run?

*Solution:* We can take the number of rotations times the circumference of $9\pi$, or the radius 4.5 times the angle in radians, $2\pi * 3 + \frac{5\pi}{4}$.

**Answer:** $9\pi \cdot \frac{29}{8} \approx 102.49$ inches

While jumping off, Rupert causes the wheel to rotate another $\frac{\pi}{6}$ radians counterclockwise.

**e**. [2 points] What angle does the dot now make with the positive horizontal axis?

**Answer:** $-\frac{\pi}{3}$ **or** $\frac{5\pi}{3}$ radians

**5**. [7 points] The parts of this problem are unrelated.

**a**. [4 points] The velocity $v$, in feet per second (ft/s), of water the leaking from a tank is proportional to the square root of the depth $d$, in feet (ft), of the water in the tank at that moment. That is, for a positive constant $k$,

$$v = k\sqrt{d}.$$

If the velocity of the water is 4 ft/s when the depth is $\frac{1}{4}$ ft, what was the velocity when the depth was 9 ft? *Show your work.*

*Solution:* Using that $v = 4$ when $d = 1/4$, we can solve

$$4 = k\sqrt{1/4} = k \cdot (1/2)$$

to find that $k = 8$.

Then when $d = 9$, we have $v = 8\sqrt{9} = 24$.

**Answer:** 24 ft/s

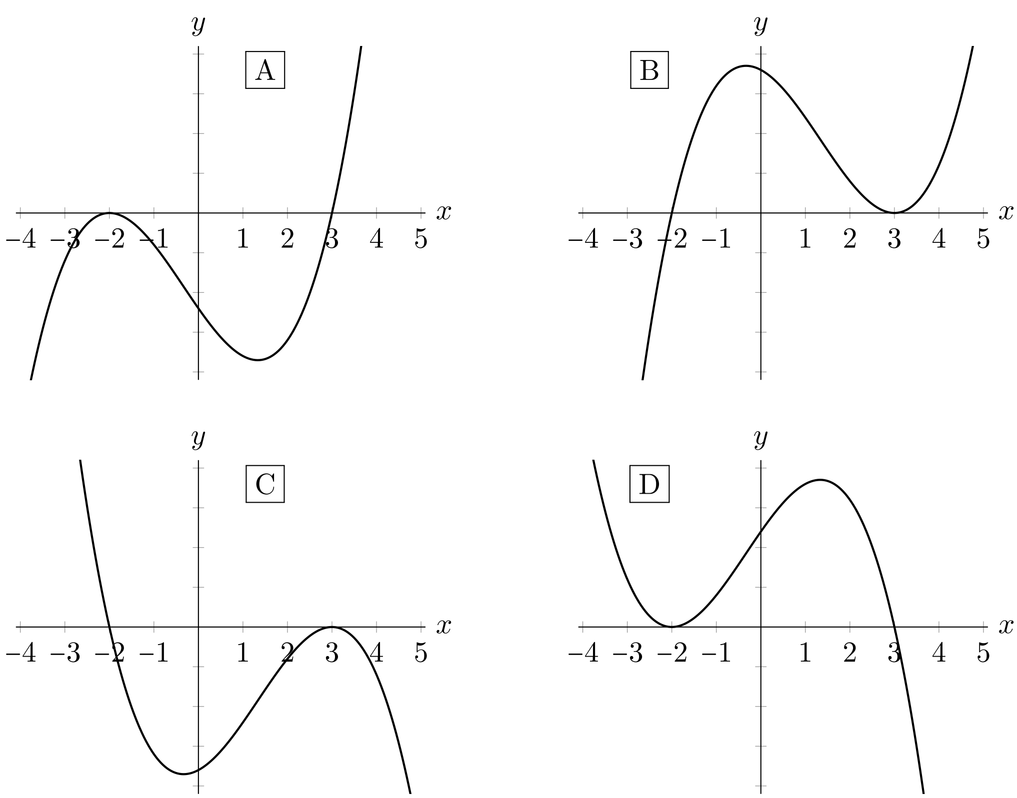
**b**. [3 points] On the lines next to each formula, write the letter corresponding to its graph.

i. $(x+2)^2(x-3)$ **A**

iii. $-(x+2)(x-3)^2$ **C**

ii. $-(x+2)^2(x-3)$ **D**

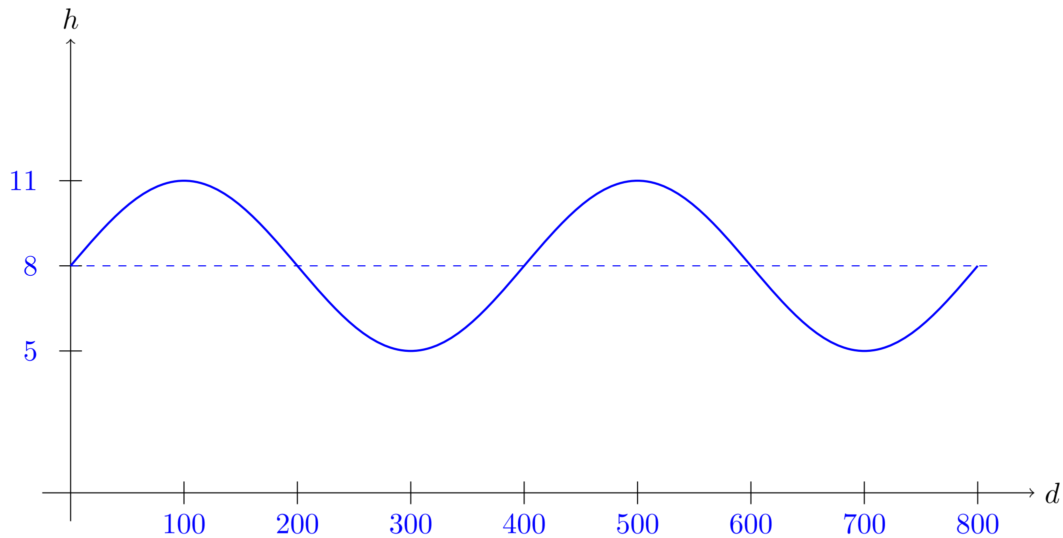
iv. $(x+2)(x-3)^2$ **B**



**6**. [14 points] On the planet of Sinosia, the number of hours of direct sunlight at a particular location varies sinusoidally throughout the year (which is not the same length as a year on Earth). In particular, the number of hours of daylight on the $d$th day is given by

$$S(d) = 8 + 3\sin\left(\frac{\pi}{200}d\right).$$

**a**. [4 points] On the axes below, sketch a graph of **two periods** of $h = S(d)$. Your second cycle should end at the $d$-value indicated by the tick mark furthest to the right. Clearly label at least two of the tick marks on the $d$-axis. On the $h$-axis, add and label at least two tick marks to indicate the maximum and minimum values of $S(d)$.




**b**. [5 points] Find the first 3 positive $d$ values for which there is 10 hours of direct sunlight. *Show your work and give answers in exact form or rounded to the nearest day.*

*Solution:* We solve the equation $S(d) = 10$ for $d$ algebraically to find the two fundamental solutions:

$$\begin{aligned} 8 + 3\sin((\pi/200)d) &= 10 \\ \sin((\pi/200)d) &= 2/3 \\ (\pi/200)d &= \arcsin(2/3) \\ \text{or } (\pi/200)d &= \pi - \arcsin(2/3). \end{aligned}$$

This leads to solutions of $d = (200/\pi)\arcsin(2/3) \approx 46$
and $d = (200/\pi)(\pi - \arcsin(2/3)) = 200 - (200/\pi)\arcsin(2/3) \approx 154$.

Then the third solution must be one period after the first, so $d = (200/\pi)\arcsin(2/3) + 400 \approx 446$.

**Answer:** $d$ = **46**, **154**, **446**

*(Problem continues on the next page.)*

On the nearby planet of Cosinia, the number of hours of direct sunlight also varies sinusoidally throughout its year.

- On the 30th day, the amount of sunlight reaches a peak of 16 hours.
- On the 80th day, the amount of sunlight is at its minimum of 4 hours.

Let $C(d)$ represent the number of hours of daylight on the $d$th day.

**c**. [1 point] How many days are there on Cosinia per year? In other words, what is the period of the function $C(d)$?

**Answer:** 100 days

**d**. [4 points] Find a formula for the function $C(d)$.

**Answer:** $C(d) = 6\cos\left(\frac{2\pi}{100}(d-30)\right) + 10$

**7.** [10 points] An amusement park is trying to decide how much to charge its visitors for admission. Consultants have predicted that, if the admission price were $\$p$, the daily number of visitors $v$ would be given by

$$v = 5{,}000 - 50\,p.$$

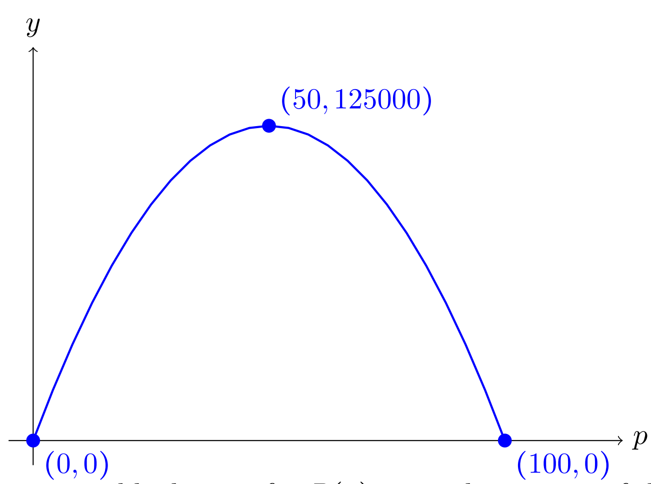
**a.** [2 points] Describe the meaning of the slope of this line in the context of this problem.

*Solution:* For each dollar increase in the cost of the park's admission ticket, 50 fewer people will visit each day.

Furthermore, the park's daily revenue from admission would be given by the quadratic function $R(p) = pv$, which can be written as

$$R(p) = -50p\,(p - 100)\,.$$

**b.** [4 points] Sketch a graph of $y = R(p)$ on the axes below. Be sure that the scale of each axis is clear, and **label the** $(p, y)$ **coordinates** of the vertex and any zeroes.




**c.** [2 points] What is a reasonable domain for $R(p)$ given the context of the problem? Briefly explain.

*Solution:* A reasonable domain is $[0, 100]$ since for any other prices, the revenue is negative, which doesn't make sense. It also doesn't make sense to have a negative price.

**d.** [2 points] What ticket price will maximize the revenue from ticket sales? Briefly explain how you know.

*Solution:* A ticket price of \$50 will maximize revenue, because this corresponds to the vertex of the parabola, which is at $p = 50$ since this is halfway between the two zeros of 0 and 100.

**8**. [10 points] *You do not need to show work in this problem, but limited partial credit may be awarded for work shown.*

**a**. [5 points] Consider the rational function:

$$q(x) = \frac{2(x-1)(x-5)}{(x+3)(x-1)}.$$

Find the following, or write NONE if none exist.

i. the $(x, y)$-**coordinates** of any hole(s): $(3, 27/28)$

ii. the **equations** of any horizontal asymptote(s): $y = 0$

iii. the **equations** of any vertical asymptotes(s): $x = -4$, $x = 1$

**b**. [5 points] Find the following limits. Each answer may be $\infty$, $-\infty$, a number, or NEI if there is not information to determine it.

i. $\lim_{x \to 0^-} \frac{1}{x} =$ $-\infty$

ii. $\lim_{x \to \infty} \ln(x) =$ $\infty$

iii. $\lim_{x \to \infty} \frac{e^x}{x^e} =$ $\infty$

iv. $\lim_{x \to \infty} \frac{2^x}{3^x} =$ $0$

v. for a quadratic function $q(x)$ and a polynomial $p(x)$ of degree 3,

$\lim_{x \to \infty} \frac{q(x)}{p(x)} =$ $0$