4. [12 points] Chump is on his yacht, enjoying his annual vacation. After finishing a bottle of Martinelli's sparkling apple cider, he tosses the empty bottle into the ocean. The trajectory of the bottle is a parabola. When the bottle is a horizontal distance of x meters away from Chump, it is $H(x)$ meters above the level of the yacht deck, where $H(x) = -x^2 + \frac{\pi}{2}x + \frac{1}{2}$.

a. [5 points] Use the method of completing the square to put $H(x)$ in vertex form. **Your answer must be exact**, and you must *show all your work, step-by-step*, to get full credit.

Solution:

$$\begin{aligned} H(x) &= -x^2 + \frac{\pi}{2}x + \frac{1}{2} \\ &= -\left(x^2 - \frac{\pi}{2}x\right) + \frac{1}{2} \\ &= -\left(x^2 - \frac{\pi}{2}x + \frac{\pi^2}{16} - \frac{\pi^2}{16}\right) + \frac{1}{2} \\ &= -\left(\left(x - \frac{\pi}{4}\right)^2 - \frac{\pi^2}{16}\right) + \frac{1}{2} \\ &= -\left(x - \frac{\pi}{4}\right)^2 + \left(\frac{\pi^2}{16} + \frac{1}{2}\right) \end{aligned}$$

$H(x) = \underline{\qquad -\left(x - \frac{\pi}{4}\right)^2 + \left(\frac{\pi^2}{16} + \frac{1}{2}\right) \qquad}$

b. [2 points] What was the maximum height of the bottle? Give your answer in exact form.

The maximum height was $\underline{\qquad \frac{\pi^2}{16} + \frac{1}{2} \text{ meters} \qquad}$ above the level of the yacht deck.

c. [5 points] Suppose the deck of the yacht is 1 meter above the surface of the ocean. What is the horizontal distance between Chump and the bottle when it hits the ocean? Leave your answer in exact form.

Solution: When the bottle hits the ocean, we have $H(x) = -1$, or

$$-\left(x - \frac{\pi}{4}\right)^2 + \left(\frac{\pi^2}{16} + \frac{1}{2}\right) = -1$$

We solve this equation for x as follows.

$$\left(x - \frac{\pi}{4}\right)^2 = \frac{\pi^2}{16} + \frac{1}{2} + 1 = \frac{\pi^2}{16} + \frac{3}{2}$$

$$x = \frac{\pi}{4} + \sqrt{\frac{\pi^2}{16} + \frac{3}{2}} \quad \text{or} \quad x = \frac{\pi}{4} - \sqrt{\frac{\pi^2}{16} + \frac{3}{2}}.$$

Only the first solution is positive, so this is the one we want.

The bottle was a horizontal distance of $\frac{\pi}{4} + \sqrt{\frac{\pi^2}{16} + \frac{3}{2}}$ meters away from Chump.