4. [9 points] Let $h(w)$ be a function, with values given in the following table:

w	1	2	4
$h(w)$	8	10	14

a. [2 points] Briefly explain why $h(w)$ could be a linear function of w, and find a linear formula for $h(w)$. For the rest of the problem, assume that $h(w)$ is linear.

Solution: We need to check that the rate of change of $h(w)$ is constant. The two rates of change we can calculate from the table are $\frac{h(2)-h(1)}{2-1} = \frac{10-8}{1} = 2$ and $\frac{h(4)-h(2)}{4-2} = \frac{14-10}{2} = 2$, and so the rate of change is constant and thus $h(w)$ could be linear. Now, we find a formula for $h(w)$. We know already that the rate of change is 2, and this is just the slope. So, we plug in a point (say $(1,8)$) to find the h-intercept:

$$\begin{aligned} h(w) &= 2w + b \\ 8 &= 2(1) + b \\ 6 &= b. \end{aligned}$$

Thus, we know that

$$h(w) = 2w + 6.$$

b. [3 points] Let $q(w)$ be a quadratic function of w, where $q(w)$ has its vertex at $(1,8)$ and passes through the w-intercept of $h(w)$. Find a formula for $q(w)$.

Solution: We use vertex form: since the vertex of $q(w)$ is at $(1,8)$, we can write

$$q(w) = a(w-1)^2 + 8.$$

We just need to find a, and to do so we plug in any other point on the graph of $q(w)$. We're told that the w-intercept of $h(w)$ is on the graph, so let's find that. We set $h(w) = 0$ and solve:

$$\begin{aligned} 2w + 6 &= 0 \\ 2w &= -6 \\ w &= -3. \end{aligned}$$

Thus the point $(-3, 0)$ is on the graph of $q(w)$. Plugging this in:

$$0 = a(-3-1)^2 + 8.$$

Solving for a, we get $a = -1/2$. So, we have

$$q(w) = -\frac{1}{2}(w-1)^2 + 8.$$

c. [1 point] Is the graph of $q(w)$ concave up or concave down?

Solution: Because $a = -1/2 < 0$, the graph is concave down. (You can also note that since the vertex of $q(w)$ is above the horizontal axis, if the graph were concave up it would "open upward" and could have no zeroes.)

d. [3 points] Find the zeroes of $q(w)$.

Solution: We set $q(w) = 0$ and solve for w:

$$\begin{aligned} -\frac{1}{2}(w-1)^2 + 8 &= 0 \\ -\frac{1}{2}(w-1)^2 &= -8 \\ (w-1)^2 &= 16 \\ w - 1 &= \pm 4 \\ w &= 1 \pm 4 \\ w &= -3, 5. \end{aligned}$$

(You can also use symmetry: we know already that $q(w)$ passes through $(-3, 0)$, and is symmetric about the vertical line $x = 1$ passing through its vertex. Since the distance from $x = -3$ to $x = 1$ is 4, the next zero occurs 4 after $x = 1$, i.e., at $x = 5$.)