6. [13 points] You're looking at buying two cars:

- Car A is worth \$30,000 initially, and the value decreases by 15% annually.
- Car B is worth \$20,000 initially, and the value also decreases exponentially. Let r be the annual growth rate. Note that r is negative.

a. [3 points] If we know that the values of Car A and Car B will be equal at some point in the future, which of the following must be true? Briefly explain your reasoning.

i. $r < -0.15$.

ii. $r > -0.15$.

iii. We do not have enough information to decide.

Solution: We must have (ii): $r > -.15$. Because Car A starts at a higher value than Car B, if Car B's value were to decay faster than Car A they would never intersect. Thus Car B's value must decay *slower* than Car A. This means that r is less negative, i.e., $r > -.15$.

b. [3 points] Suppose that r is some value so that the cars do eventually become the same price, and then r increases (so r gets closer to 0) and everything else stays the same. Will the time it takes for the two cars to become equal in value increase or decrease? Briefly explain your reasoning.

Solution: Decrease: As r increases (gets closer to 0), the rate at which Car B is declining in value *slow*. Since Car A starts off more valuable than Car B, the slower Car B declines the less Car A has to drop in value before they're equal in value. So, the point of intersection will happen sooner, and thus the time decreases.

c. [4 points] Let t be the number of years from now when the two cars are equal in value. Find t (in exact form). Your answer may contain r.

Solution: The value of Car A is given by $30000(.85)^t$, and the value of Car B by $20000(1+r)^t$. Setting these equal and isolating t in the exponent, we have:

$$20000(1+r)^t = 30000(.85)^t$$
$$\frac{(1+r)^t}{.85^t} = \frac{30000}{20000}$$
$$\left(\frac{1+r}{.85}\right)^t = \frac{3}{2}.$$

Now we take log of both sides (we could also use ln) and use log rules:

$$\log\left(\left(\frac{1+r}{.85}\right)^t\right) = \log\left(\frac{3}{2}\right)$$
$$t\log\left(\frac{1+r}{.85}\right) = \log\left(\frac{3}{2}\right)$$
$$t = \frac{\log\left(\frac{3}{2}\right)}{\log\left(\frac{1+r}{.85}\right)}.$$

You can write your answer equally correctly as

$$t = \frac{\log 3 - \log 2}{\log(1+r) - \log(.85)},$$

or with ln in place of log.

d. [3 points] If the cars will be equal in value in 10 years, find r (in exact form).

Solution: We start the same way as part (c): we set the two cars value equal, but with $t = 10$ plugged in. We then solve for r:

$$20000(1+r)^{10} = 30000(.85)^{10}$$
$$\left(\frac{1+r}{.85}\right)^{10} = \frac{3}{2}$$

Now, we take tenth roots:

$$\frac{1+r}{.85} = \left(\frac{3}{2}\right)^{1/10}$$
$$1+r = .85\cdot\left(\frac{3}{2}\right)^{1/10}$$
$$r = .85\cdot\left(\frac{3}{2}\right)^{1/10} - 1.$$

(You can also do part (d) by taking your answer for part (c), plugging in $t = 10$, and solving for r.)