2. [11 points] The population of Detroit in 1970 was 1.51 million and by 1990 it was 1.03 million. In parts (a)–(c) below, use the following variable definitions:

- Let D be the population of Detroit in **millions of people**.
- Let t be **years since 1970**.

a. [3 points] If we assume the population of Detroit decreased **exponentially** as a function of time, find a formula for $D = E(t)$.

Show all work. Leave constants in exact form, or rounded to at least two decimal places.

Solution: The fact that the population was 1.51 million people in 1970 tells us that $D(0) = 1.51$. Similarly, we have $D(20) = 1.03$ since the population was 1.03 million people in 1990, 20 years after 1970.

If we assume $D = E(t)$ is an exponential function, then it can be written in the form $E(t) = ab^t$ for some constants a and b. We need to solve for a and b. Using the information above, we have

$$E(0) = ab^0 = a = 1.51$$

and

$$E(20) = ab^{20} = 1.03$$

The first equation gives us that $a = 1.51$. Substituting this value of a into the second equation gives us

$$\begin{aligned} 1.51b^{20} &= 1.03 \\ b^{20} &= \frac{1.03}{1.51} \\ b &= \pm\sqrt[20]{\frac{1.03}{1.51}} \end{aligned}$$

Since the growth factor of an exponential function must be positive, we have $b = \sqrt[20]{\frac{1.03}{1.51}} \approx 0.9811$. Therefore, $E(t) = 1.51\left(\sqrt[20]{\frac{1.03}{1.51}}\right)^t$

$D = E(t) =$ $1.51\left(\sqrt[20]{\frac{1.03}{1.51}}\right)^t$

b. [2 points] Is the graph of $D = E(t)$ concave up or concave down or neither? Provide a small sketch or explain *briefly* how you know.

(Circle One) $\boxed{\text{CONCAVE UP}}$ CONCAVE DOWN

Explanation or Sketch:

Solution: An exponential function of the form $P(t) = ab^t$ where $a > 0$ is always concave up.

c. [3 points] If we assume, instead, that the population of Detroit decreased **linearly** as a function of time, find a formula for $D = L(t)$.

Show all work. Leave constants in exact form, or rounded to at least two decimal places.

Solution: The graph of $L(t)$ should go through the points $(0, 1.51)$ and $(20, 1.03)$ as we saw in part (a). The slope of such a linear function is $\frac{1.03-1.51}{20-0} = \frac{-0.48}{20} = -0.024$. Since we already have the value of $L(0)$ is 1.51, we know the vertical intercept is 1.51. Therefore $L(t) = -0.024t + 1.51$. Alternatively, we could use the point $(20, 1.03)$ and write the answer using point-slope form: $L(t) = -0.024(t - 20) + 1.03$.

$D = L(t) =$ $-0.024(t-20) + 1.03 = -0.024t + 1.51$

d. [3 points] Fort Myers, Florida had a population of 105,260 at the beginning of 2022, which grew by 6.82% over the next year. If the population continues to grow exponentially, how large will the city be in 2030?

Show all work. Leave answer in exact form, or rounded to the nearest whole number.

Solution: Let $F(t)$ be the population of Fort Myers, Florida t years since the start of 2022. If we assume the population is growing exponentially, then $F(t)$ has the form $F(t) = ab^t$. The fact that the population grew by 6.82% over one year (note that this agrees with the units of t) tells us that $r = 0.0682$, so $b = 1.0682$. Since we know $F(0) = 105,260$, we also have that the initial quantity is $a = 105,260$.
More explicitly, we could also solve for these values using the formula for $F(t)$: We are given that $F(0) = 105,260$ and $F(1) = 1.0682(F(0))$.
Therefore,

$$F(0) = ab^0 = a = 105,260$$

and

$$F(1) = ab = 1.0682(105,260)$$

The first equation tells us that $a = 105,260$. Substituting this into the second equation gives us that

$$105,260b = 1.0682(105,260)$$
$$b = 1.0682$$

So, $F(t) = 105,260(1.0682)^t$. The year 2030 corresponds to $t = 8$ since 2030 is 8 years after 2022. We can compute $F(8) = 105,260(1.0682)^8 \approx 178437$ to see that the population would be $105,260(1.0682)^8 \approx 178,437$ people in 2030.

Population in 2030: $105,260(1.0682)^8 \approx 178,437$ **people**