8. [14 points] The owner of a restaurant has a budget to buy up to 15 hours of advertising time on the radio. She predicts that her profits $P(x)$, in thousands of dollars, when she buys x minutes of advertising on the radio for her restaurant is given by:

$$P(x) = -3x^2 + 40x + 100 \qquad \text{for} \qquad 0 \le x \le 15.$$

a. [5 points] Write the formula of $P(x)$ in vertex form by completing the square. Show all your work step-by-step to receive full credit.

Solution:

$$\begin{aligned} P(x) &= -3x^2 + 40x + 100 \\ &= -3\left(x^2 - \frac{40}{3}x\right) + 100 \\ &= -3\left(x^2 - \frac{40}{3}x + \left(-\frac{20}{3}\right)^2 - \left(-\frac{20}{3}\right)^2\right) + 100 \\ &= -3\left(\left(x - \frac{20}{3}\right)^2 - \left(\frac{20}{3}\right)^2\right) + 100 \\ &= -3\left(x - \frac{20}{3}\right)^2 + 3\left(\frac{20}{3}\right)^2 + 100 \\ &= -3\left(x - \frac{20}{3}\right)^2 + \frac{700}{3} \end{aligned}$$

b. [3 points] Find the practical domain and range of the function $P(x)$. Your answers must be written in **exact form** or accurate up to the first two decimals. Use inequalities or interval notation.

Solution: Domain: $[0, 15]$ Range: $\left[25, \frac{700}{3}\right]$

The statement of the problem has been included below for your convenience.

The owner of a restaurant has a budget to buy up to 15 hours of advertising time on the radio. She predicts that her profits $P(x)$, in thousands of dollars, when she buys x minutes of advertising on the radio for her restaurant is given by:

$$P(x) = -3x^2 + 40x + 100 \qquad \text{for} \qquad 0 \leq x \leq 15.$$

c. [3 points] What should be the minimum amount of radio advertising time the owner has to buy if she wants to obtain a profit of one hundred fifty thousand dollars?
Your answer should be obtained **algebraically** and it must be in **exact form** or accurate up to the first two decimals. Include units. Show all your work.

Solution:

$$-3x^2 + 40x + 100 = 150$$
$$-3x^2 + 40x - 50 = 0$$
$$x = \frac{-40 \pm \sqrt{(40)^2 - 4(-3)(-50)}}{-6} = \frac{40 \pm \sqrt{1000}}{6}$$
$$x = \frac{40 - \sqrt{1000}}{6} \quad \text{hours}$$

d. [3 points] Find the average rate of change of the function $P(x)$ for $10 \leq x \leq 15$. Include units. Show all your work.

Solution:

$$\frac{\Delta P}{\Delta x} = \frac{P(15) - P(10)}{15 - 10} = \frac{25 - 200}{5} = -\frac{175}{5} = -35 \text{ thousands of dollars per hour}$$