6. [14 points] After a day of work on the farm, Percy likes to toss corn cobs from the second story window of the barn to the ground. On one toss, the corn cob follows a parabolic path $h(x) = -x^2 + bx + c$ where $h(x)$ is the height of the cob above the ground, in feet, when it is a horizontal distance x feet from the barn. The numbers b and c are constants.

a. [3 points] Interpret the vertical intercept of $h(x)$ in the context of this problem.

Solution: The vertical intercept of $h(x)$ is the height of the window above the ground (in feet).

b. [4 points] If the window is 9 feet from the ground, and the cob hits the ground 9 feet from the barn, find the values of the constants b and c. Show your work.

$b =$ 8

$c =$ 9

Solution: The information in the problem tells us that $(0, 9)$ and $(9, 0)$ are both on path of the cob before the first bounce. We see $c = 9$ immediately using the point $(0, 9)$, so we just need to find b in $h(x) = -x^2 + bx + 9$. Using 9 for x and 0 for $h(x)$, $0 = -81 + 9b + 9$, so $b = 8$.

c. [4 points] After the cob bounces, it follows a path given by $p(x) = -\frac{1}{3}x^2 + 8x - 45$ where $p(x)$ is the height of the cob above the ground, in feet, when it is a horizontal distance x feet from the barn. By completing the square, find the maximum height the cob achieves after it bounces. You must show all steps of your calculation.

maximum height = 3 feet

Solution:

$$\begin{aligned} p(x) &= -\frac{1}{3}x^2 + 8x - 45 \\ &= -\frac{1}{3}(x^2 - 24x) - 45 \\ &= -\frac{1}{3}(x^2 - 24x + 144) - 45 + 48 \\ &= -\frac{1}{3}(x - 12)^2 + 3 \end{aligned}$$

So 3 feet off the ground is the maximum height achieved by the cob.

d. [3 points] Find the distance the cob is from the barn when it hits the ground for the second time. Show your work. Hint: Use the quadratic formula.

distance = 15 feet

Solution: The quadratic formula gives

$$x = \frac{-8 \pm \sqrt{64 - 4(-1/3)(-45)}}{-2/3} = \frac{-8 \pm 2}{-2/3}.$$

So $x = 9, 15$, but $x = 9$ is the location of the first bounce, so the second bounce must be at $x = 15$.