5. [12 points] Jack is starting a business teaching others to paint. He has come up with the following pricing plan.

- For each lesson, a client has to pay a flat fee of \$6 to cover the cost of the art supplies they will use.
- He charges \$2 per minute for the first 60 minutes of the lesson.
- He charges \$0.50 per minute for each minute after that.
- Each lesson lasts at most 120 minutes.

Let $C(m)$ be the amount of money he charges for a lesson that is m minutes long.

a. [2 points] Evaluate $C(70)$.

Solution:
$C(70)$ is the amount of money (measured in dollars) that Jack charges for a lesson that is 70 minutes long. For such a lesson, Jack charges a flat fee of \$6 to cover the cost of art supplies, \$2 per minute for the first 60 minutes of the lesson, and \$0.50 per minute for the final ten minutes of the lesson. Therefore,

$$\boxed{C(70) = 6 + 2 \cdot 60 + 0.50 \cdot 10 = 6 + 120 + 5 = 131}$$

That is, Jack charges \$131 for a 70 minute lesson.

b. [6 points] Find a formula for $C(m)$. Use standard piecewise function notation:

$$C(m) = \begin{cases} \\ \\ \end{cases}$$

Solution:
Since a lesson can last from 0 to 120 minutes long, the domain of the function $C(m)$ is given by the inequality $0 < m \leq 120$. Since Jack charges different rates for the first 60 minutes of a lesson and any remaining time afterwards, we will split this domain into two pieces: $0 < m \leq 60$, and $60 < m \leq 120$.
If a lesson is $0 < m \leq 60$ minutes long, then Jack charges a flat fee of \$6 to cover the cost of art supplies as well as \$2 per minute for all m minutes. Thus for $0 < m \leq 60$,

$$C(m) = 6 + 2m.$$

On the other hand, if the lesson is $60 < m \leq 120$ minutes long, then Jack charges a flat fee of \$6 to cover the cost of art supplies, \$2 per minute for the first 60 minutes, and \$0.50 per minute for the remaining $m - 60$ minutes. Thus

$$C(m) = 6 + 2 \cdot 60 + 0.50(m - 60) = 96 + 0.50m.$$

This gives the following piecewise-defined formula for $C(m)$:

$$\boxed{C(m) = \begin{cases} 6 + 2m, & \text{if } 0 < m \leq 60 \\ 96 + 0.50m, & \text{if } 60 < m \leq 120. \end{cases}}$$

c. [4 points] The function $d = C(m)$, where d is the cost (in dollars) of a painting lesson that lasts m minutes, is invertible. Write a formula for its inverse $C^{-1}(d)$ using standard piecewise function notation.

Solution:
In order to find a formula for the inverse of $C(m)$, we must invert the formulas given for $C(m)$ above. Since these are linear functions, this can be done algebraically as follows:

$$\begin{aligned} d &= 6 + 2m \\ 2m &= d - 6 \\ m &= 0.5d - 3 \end{aligned} \qquad\qquad \begin{aligned} d &= 96 + 0.50m \\ 0.50m &= d - 96 \\ m &= 2d - 192 \end{aligned}$$

We must also find the domains on which these formulas are valid. In order to do this, remember that inverting a function switches its domain and range. The same thing is true for the pieces of a piecewise defined function.
The formula $C(m) = 6 + 2m$ is valid on the domain $0 < m \leq 60$. On this domain, this formula has range $6 < d \leq 126$. Thus the formula $C^{-1}(d) = 0.5d - 3$ is valid on the interval $6 < d \leq 126$.
Similarly, the formula $C(d) = 96 + 0.50m$ is valid on the domain $60 < m \leq 120$. On this domain, this formula has range $126 < d \leq 156$. Thus the formula $C^{-1}(d) = 2d - 192$ is valid on the interval $126 < d \leq 156$. In summary, the inverse function $m = C^{-1}(d)$ has piecewise-defined formula

$$\boxed{C^{-1}(d) = \begin{cases} 0.5d - 3, & \text{if } 6 < d \leq 126 \\ 2d - 192, & \text{if } 126 < d \leq 156. \end{cases}}$$