5. [12 points] Jack is starting a business teaching others to paint. He has come up with the following pricing plan.

- For each lesson, a client has to pay a flat fee of $6 to cover the cost of the art supplies they will use.
- He charges $2 per minute for the first 60 minutes of the lesson.
- He charges $0.50 per minute for each minute after that.
- Each lesson lasts at most 120 minutes.

Let $C(m)$ be the amount of money he charges for a lesson that is $m$ minutes long.

a. [2 points] Evaluate $C(70)$.

**Solution:**

$C(70)$ is the amount of money (measured in dollars) that Jack charges for a lesson that is 70 minutes long. For such a lesson, Jack charges a flat fee of $6 to cover the cost of art supplies, $2 per minute for the first 60 minutes of the lesson, and $0.50 per minute for the final ten minutes of the lesson. Therefore,

$$C(70) = 6 + 2 \cdot 60 + 0.50 \cdot 10 = 6 + 120 + 5 = 131$$

That is, Jack charges $131 for a 70 minute lesson.
b. [6 points] Find a formula for $C(m)$. Use standard piecewise function notation:

$$C(m) = \begin{cases} 
6 + 2m, & \text{if } 0 < m \leq 60 \\
96 + 0.50m, & \text{if } 60 < m \leq 120.
\end{cases}$$

\textbf{Solution:}

Since a lesson can last from 0 to 120 minutes long, the domain of the function $C(m)$ is given by the inequality $0 < m \leq 120$. Since Jack charges different rates for the first 60 minutes of a lesson and any remaining time afterwards, we will split this domain into two pieces: $0 < m \leq 60$, and $60 < m \leq 120$.

- If a lesson is $0 < m \leq 60$ minutes long, then Jack charges a flat fee of $6 to cover the cost of art supplies as well as $2 per minute for all $m$ minutes. Thus for $0 < m \leq 60$,

  $$C(m) = 6 + 2m.$$ 

- On the other hand, if the lesson is $60 < m \leq 120$ minutes long, then Jack charges a flat fee of $6 to cover the cost of art supplies, $2 per minute for the first 60 minutes, and $0.50 per minute for the remaining $m - 60$ minutes. Thus

  $$C(m) = 6 + 2 \cdot 60 + 0.50 (m - 60) = 96 + 0.50m.$$ 

This gives the following piecewise-defined formula for $C(m)$:

$$C(m) = \begin{cases} 
6 + 2m, & \text{if } 0 < m \leq 60 \\
96 + 0.50m, & \text{if } 60 < m \leq 120.
\end{cases}$$
c. [4 points] The function \( d = C(m) \), where \( d \) is the cost (in dollars) of a painting lesson that lasts \( m \) minutes, is invertible. Write a formula for its inverse \( C^{-1}(d) \) using standard piecewise function notation.

**Solution:**

In order to find a formula for the inverse of \( C(m) \), we must invert the formulas given for \( C(m) \) above. Since these are linear functions, this can be done algebraically as follows:

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\begin{align*}
    d &= 6 + 2m \\
    2m &= d - 6 \\
    m &= 0.5d - 3 \\
    d &= 96 + 0.50m \\
    0.50m &= d - 96 \\
    m &= 2d - 192
\end{align*}
\]

We must also find the domains on which these formulas are valid. In order to do this, remember that inverting a function switches its domain and range. The same thing is true for the pieces of a piecewise defined function.

The formula \( C(m) = 6 + 2m \) is valid on the domain \( 0 < m \leq 60 \). On this domain, this formula has range \( 6 < d \leq 126 \). Thus the formula \( C^{-1}(d) = 0.5d - 3 \) is valid on the interval \( 6 < d \leq 126 \).

Similarly, the formula \( C(d) = 96 + 0.50m \) is valid on the domain \( 60 < m \leq 120 \). On this domain, this formula has range \( 126 < d \leq 156 \). Thus the formula \( C^{-1}(d) = 2d - 192 \) is valid on the interval \( 126 < d \leq 156 \). In summary, the inverse function \( m = C^{-1}(d) \) has piecewise-defined formula

\[
C^{-1}(d) = \begin{cases} 
0.5d - 3, & \text{if } 6 < d \leq 126 \\
2d - 192, & \text{if } 126 < d \leq 156.
\end{cases}
\]