6. [15 points] Scientists discover a new island in Lake Michigan and begin studying its animals. The island has both lizards and crows when they arrive, and they accidentally leave some mice on the island after discovering it.

- 5 thousand lizards live on the island when they discover it, but the population is decreasing at a rate of 5% per year.
- Half a year after the island is discovered, the population of mice has grown to 2.3 times the initial population, and appears to be growing exponentially.
- The population of crows, in thousands, t years after the island is discovered, can be modeled by $C(t) = 4e^{0.06t-1}$.

In the following problems, leave your answer in **exact form** and show every step of your work.

a. [3 points] Find a formula for $L(t)$, the number of lizards on the island, in thousands, t years after the island is discovered.

Solution:
$L(t)$ is the number of lizards on the island (measured in thousands) t years after the island is discovered. $L(t)$ is decaying exponentially with an initial value of 5 and a yearly decay rate of 5%. Thus $L(t)$ has formula

$$\boxed{L(t) = 5(1-0.05)^t}$$

b. [3 points] How long does it take for the population of mice to reach 10 times the initial population?

Solution:
Let $M(t)$ denote the number of mice living on the island t years after the island is discovered. As an exponential function, we can write $M(t) = ab^t$ for some real numbers a and b. With the information given, we cannot hope to find the initial value a, but we can find the growth factor b.
Specifically, in half a year, the population of mice has grown exponentially to 2.3 times its original size. This gives an equation $M(0.5) = 2.3 \cdot M(0)$, which we can solve to find the growth factor b:

$$\begin{aligned} ab^{0.5} &= 2.3 \cdot ab^0 \\ a\sqrt{b} &= 2.3a \\ \sqrt{b} &= 2.3 \\ b &= (2.3)^2 \end{aligned}$$

Thus $M(t) = a(2.3)^{2t}$. We now want to find how long it will take for the population of mice to reach 10 times its original size. This condition is represented by the equation $M(t) = 10 \cdot M(0)$, which we can solve for t:

$$\begin{aligned} a(2.3)^{2t} &= 10 \cdot a(2.3)^{2\cdot 0} \\ (2.3)^{2t} &= 10 \\ \log\left(2.3^{2t}\right) &= \log(10) \\ 2t \cdot \log(2.3) &= 1 \\ t &= \frac{1}{2\log(2.3)} \end{aligned}$$

Therefore, it will take $\frac{1}{2\log(2.3)}$ years for the population of mice to reach 10 times its initial size.

c. [2 points] What is the vertical intercept of $C(t)$? Interpret the meaning of this number in the context of the problem.

Solution:
The population of crows (measured in thousands) t years after the island is discovered is given by the formula $C(t) = 4e^{0.06t-1}$. This formula for $C(t)$ is not in standard or continuous growth form, so we cannot immediately read off the vertical intercept from the formula. Instead, we must compute

$$C(0) = 4e^{0.06\cdot 0-1} = 4e^{-1} = \frac{4}{e}.$$

Thus $C(t)$ has vertical intercept $\boxed{\left(0, \frac{4}{e}\right)}$.
This tells us that the number of crows when the island is discovered is $\frac{4}{e}$ thousand.

d. [2 points] By what percentage does the population of crows increase in a year?

Solution:
Let's put the formula $C(t) = 4e^{0.06t-1}$ in standard form:

$$C(t) = 4e^{0.06t-1} = \frac{4e^{0.06t}}{e} = \left(\frac{4}{e}\right)\left(e^{0.06}\right)^t$$

So $C(t)$ has yearly growth factor $e^{0.06}$. This implies that the yearly percent growth rate is $100\left(e^{0.06}-1\right)\%$; that is, the population of crows grows by $\boxed{100\left(e^{0.06}-1\right)\%}$ every year.

e. [5 points] When will there be the same number of lizards and crows on the island?

Solution:
Recall that the populations of lizards and crows (measured in thousands) t years after the island is discovered are given by the formulas $L(t) = 5(1-0.05)^t$ and $C(t) = 4e^{0.06t-1}$, respectively. We can use these formulas to solve the equation $C(t) = L(t)$ for t:

$$\begin{aligned}
5(1-0.05)^t &= 4e^{0.06t-1} \\
\ln\left(5 \cdot 0.95^t\right) &= \ln\left(4e^{0.06t-1}\right) \\
\ln(5) + t \cdot \ln(0.95) &= \ln(4) + 0.06t - 1 \\
t \cdot \ln(0.95) - 0.06t &= \ln(4) - \ln(5) - 1 \\
t\left(\ln(0.95) - 0.06\right) &= \ln(4) - \ln(5) - 1 \\
t &= \frac{\ln(4) - \ln(5) - 1}{\ln(0.95) - 0.06}
\end{aligned}$$

Therefore, the populations of lizards and crows on the island will agree $\boxed{\dfrac{\ln(4) - \ln(5) - 1}{\ln(0.95) - 0.06}}$ years after the island is discovered.