7. [14 points] In music, the *pitch* of a tone, P, measured in *cents*, is a function of the tone's frequency, f, measured in hertz. The pitch is defined to be

$$P = 6000 + k \ln\left(\frac{f}{f_0}\right)$$

where f_0 is the frequency of a tone called "middle C", and k is a constant.

a. [2 points] What is the pitch of "middle C"? (*Remember to include units.*)

Solution: The frequency of "middle C" is f_0, so the pitch of "middle C" is $6000 + k \ln\left(\frac{f_0}{f_0}\right) = 6000 + k\ln(1) = 6000 + k(0) = 6000$.

Answer: 6000 **cents**

b. [3 points] If the frequency of one tone is two times the frequency of middle C, then the pitch of that tone is 7200 cents. Use this information to find the *exact* value of k. Then give an approximation of k rounded to the nearest 0.1.

Solution: If the frequency of a tone is two times the frequency f_0 of middle C, then its frequency is $2f_0$. So we have $7200 = 6000 + k \ln\left(\frac{2f_0}{f_0}\right) = 6000 + k\ln(2)$. Solving for k we find $1200 = k\ln(2)$ so $k = 1200/\ln(2) \approx 1731.2$.

Exact value of k: $\dfrac{1200}{\ln 2}$ **Approximation:** 1731.2

Use the *approximation* of k you found in part (b) to answer the questions below.
(If you were unable to answer part (b), leave your answers below in terms of k.)

c. [4 points] Let P_1 and P_2 represent the pitches of tones of frequency f_1 and f_2, respectively. Find a formula for the difference in pitches, $P_2 - P_1$, in terms of the two frequencies f_1 and f_2. *Simplify your answer; your formula should* **not** *involve* f_0.

Solution: We have

$$\begin{aligned} P_2 - P_1 &= \left(6000 + k\ln\left(\frac{f_2}{f_0}\right)\right) - \left(6000 + k\ln\left(\frac{f_1}{f_0}\right)\right) = k\ln\left(\frac{f_2}{f_0}\right) - k\ln\left(\frac{f_1}{f_0}\right) \\ &= k\left(\ln\left(\frac{f_2}{f_0}\right) - \ln\left(\frac{f_1}{f_0}\right)\right) = k\ln\left(\frac{f_2/f_0}{f_1/f_0}\right) = k\ln\left(\frac{f_2}{f_1}\right) \end{aligned}$$

Using our approximation of k from above, we find that $P_2 - P_1$ is approximately $1731.2\ln\left(\frac{f_2}{f_1}\right)$.

Answer: $P_2 - P_1 = k\ln\left(\dfrac{f_2}{f_1}\right) \approx 1731.2\ln\left(\dfrac{f_2}{f_1}\right)$

d. [5 points] The tone called "A above middle C" has a frequency of 440 hertz. Find the frequency of the tone whose pitch is 400 cents higher than the pitch of "A above middle C." (*Remember to include units.*)

Solution: Let f denote the frequency of the tone whose pitch is 400 cents higher than "A above middle C." Using the solution to part (c), we know that $400 = k\ln\left(\frac{f}{440}\right)$. Dividing through by k gives $\ln\left(\frac{f}{440}\right) = 400/k$. Exponentiating both sides gives $\dfrac{f}{440} = e^{400/k}$. We then find that $f = 440e^{400/k} \approx 554.37$ Hz.

Answer: **About** 554.37 **Hz**