**3.** [15 points] A scientist is observing two different ant colonies under different experimental conditions. From her data, it looks like

- Colony A's population increases by 10% every two hours.
- Colony B's population decreases by 7% every hour.

**a.** [1 point] By what factor is Colony A's population multiplied each hour? *Give your answer in exact form or rounded to two decimal places.*

a factor of: $\sqrt{1.1} \approx 1.049$

**b.** [2 points] What is the *continuous* decay rate of Colony B per hour as a percentage? *Give your answer in exact form or rounded to two decimal places.*

*Solution:* A 7% decay rate means we have a growth factor of 0.93. So we need to find the value of $k$ such that $e^k = 0.93$. This is equivalent to $k = \ln(0.93) \approx -.07257$. Since we are asked for the *decay* rate, we should give the positive version, and give it as a percent: $|\ln(0.93)| \times 100$ % or$-\ln(0.93) \times 100$ % or 7.257%.

$-\ln(0.93) \times 100 \approx 7.257$ %

**c.** [2 points] How long will it take for Colony B to reach 25% of its original size? *Show all work. Give your answer in exact form or rounded to two decimal places.*

*Solution:* We need to solve for $t$:

$$\begin{aligned} Q_0 0.93^t &= 0.25 Q_0 \\ 0.93^t &= 0.25 \\ \ln(0.93^t) &= \ln(0.25) \\ t\ln(0.93) &= \ln(0.25) \\ t &= \frac{\ln 0.25}{\ln 0.93} \end{aligned}$$

$\frac{\ln 0.25}{\ln 0.93} \approx 19.103$ hours

**d.** [4 points] If Colony A starts with 1000 ants and Colony B starts with 10,000 ants, after how many hours will the colonies have equal populations? *Show all work. Give your answer in exact form or rounded to two decimal places.*

*Solution:* A formula for the number of ants in Colony A is $1000 \cdot (\sqrt{1.1})^t$. A formula for the number of ants in Colony B is $10000 \cdot 0.93^t$. We want to find the value of $t$ that makes these two functions equal. That is, we need to solve the following for $t$:

$$\begin{aligned}
1000 \cdot (\sqrt{1.1})^t &= 10000 \cdot 0.93^t \\
\left(\frac{\sqrt{1.1}}{0.93}\right)^t &= 10 \\
\log\left(\frac{\sqrt{1.1}}{0.93}\right)^t &= \log 10 \\
t \log\left(\frac{\sqrt{1.1}}{0.93}\right) &= 1 \\
t &= \frac{1}{\log\left(\frac{\sqrt{1.1}}{0.93}\right)} \approx 19.152
\end{aligned}$$

Another way to solve the same starting equation is shown below. Note that the final answers will look different. But we can either use a calculator or log identities to see that they are actually equivalent.

$$\begin{aligned}
1000 \cdot (\sqrt{1.1})^t &= 10000 \cdot 0.93^t \\
\ln(1000 \cdot (\sqrt{1.1})^t) &= \ln(10000 \cdot 0.93^t) \\
\ln(1000) + \ln(\sqrt{1.1})^t) &= \ln(10000) + \ln(0.93^t) \\
\ln(1000) + t\ln(\sqrt{1.1})) &= \ln(10000) + t\ln(0.93) \\
t\ln(\sqrt{1.1})) - t\ln(0.93) &= \ln(10000) - \ln(1000) \\
t\left(\ln(\sqrt{1.1}) - \ln(0.93)\right) &= \ln(10000) - \ln(1000) \\
t &= \frac{\ln(10000) - \ln(1000)}{\ln(\sqrt{1.1}) - \ln(0.93)} \approx 19.152
\end{aligned}$$

$\frac{1}{\log\left(\frac{\sqrt{1.1}}{0.93}\right)} \approx 19.152$ hours

*(Problem continues on the next page.)*

The scientist now observes two additional different ant colonies. From her data, it looks like

- Colony C's population doubles for the first time after 2.5 hours; doubles again 1.5 hours after that; then doubles a third time 1 hour after that.
- Colony D's population is given by $P = D(t) = 1200 - 300e^{-0.11t}$, where $P$ is the number of ants and $t$ is measured in hours since the experiment started.

**e**. [2 points] Is Colony C growing exponentially? Circle your answer below. If YES, find its growth factor. If NO, explain why not.

YES NO

**Explanation or Growth Factor:**

*Solution:*
NO.

Any exponentially growing function should have a *constant* doubling time. Since the doubling time of this function is changing, it cannot be growing exponentially.

**f**. [4 points] Find a general formula $D^{-1}(P)$ and explain what that function means. *Show all work.*

*Solution:* We need to solve $P = 1200 - 300e^{-0.11t}$ for $t$, which will give us $t$ as a function of $P$— in other words, our inverse function.

$$\begin{aligned} P &= 1200 - 300e^{-0.11t} \\ P - 1200 &= -300e^{-0.11t} \\ \frac{P-1200}{-300} = \frac{1200-P}{300} &= e^{-0.11t} \\ \ln\left(\frac{1200-P}{300}\right) &= -0.11t \\ t &= \frac{1}{-0.11}\ln\left(\frac{1200-P}{300}\right) \end{aligned}$$

$D^{-1}(P) =$ $\frac{1}{-0.11}\ln\left(\frac{1200-P}{300}\right)$

**Meaning of $D^{-1}(P)$:**

*Solution:* $D^{-1}(P)$ gives us the number of hours after the experiment started at which there are $P$ ants in Colony D.