3. [15 points] A scientist is observing two different ant colonies under different experimental conditions. From her data, it looks like

- Colony A’s population increases by 10% every two hours.
- Colony B’s population decreases by 7% every hour.

a. [1 point] By what factor is Colony A’s population multiplied each hour? Give your answer in exact form or rounded to two decimal places.

A factor of: \( \sqrt{1.1} \approx 1.049 \)

b. [2 points] What is the continuous decay rate of Colony B per hour as a percentage? Give your answer in exact form or rounded to two decimal places.

Solution: A 7% decay rate means we have a growth factor of 0.93. So we need to find the value of \( k \) such that \( e^k = 0.93 \). This is equivalent to \( k = \ln(0.93) \approx -0.07257 \). Since we are asked for the decay rate, we should give the positive version, and give it as a percent: \( |\ln(0.93)| \times 100 \% \) or \( -\ln(0.93) \times 100 \% \) or 7.257%.

\[ - \ln(0.93) \times 100 \approx 7.257 \% \]

c. [2 points] How long will it take for Colony B to reach 25% of its original size? Show all work. Give your answer in exact form or rounded to two decimal places.

Solution: We need to solve for \( t \):

\[
Q_0 0.93^t = 0.25Q_0 \\
0.93^t = 0.25 \\
\ln(0.93^t) = \ln(0.25) \\
t \ln(0.93) = \ln(0.25) \\
t = \frac{\ln 0.25}{\ln 0.93} \\
t \approx 19.103 \text{ hours}
\]

d. [4 points] If Colony A starts with 1000 ants and Colony B starts with 10,000 ants, after how many hours will the colonies have equal populations? Show all work. Give your answer in exact form or rounded to two decimal places.
Solution: A formula for the number of ants in Colony A is 1000 \cdot (\sqrt{1.1})^t. A formula for the number of ants in Colony B is 10000 \cdot 0.93^t. We want to find the value of \( t \) that makes these two functions equal. That is, we need to solve the following for \( t \):

\[
1000 \cdot (\sqrt{1.1})^t = 10000 \cdot 0.93^t
\]

\[
\left( \frac{\sqrt{1.1}}{0.93} \right)^t = 10
\]

\[
\log\left( \frac{\sqrt{1.1}}{0.93} \right)^t = \log 10
\]

\[
t \log \left( \frac{\sqrt{1.1}}{0.93} \right) = 1
\]

\[
t = \frac{1}{\log \left( \frac{\sqrt{1.1}}{0.93} \right)} \approx 19.152
\]

Another way to solve the same starting equation is shown below. Note that the final answers will look different. But we can either use a calculator or log identities to see that they are actually equivalent.

\[
1000 \cdot (\sqrt{1.1})^t = 10000 \cdot 0.93^t
\]

\[
\ln(1000 \cdot (\sqrt{1.1})^t) = \ln(10000 \cdot 0.93^t)
\]

\[
\ln(1000) + \ln((\sqrt{1.1})^t) = \ln(10000) + \ln(0.93^t)
\]

\[
\ln(1000) + t \ln(\sqrt{1.1}) = \ln(10000) + t \ln(0.93)
\]

\[
t \ln(\sqrt{1.1}) - t \ln(0.93) = \ln(10000) - \ln(1000)
\]

\[
t \left( \ln(\sqrt{1.1}) - \ln(0.93) \right) = \ln(10000) - \ln(1000)
\]

\[
t = \frac{\ln(10000) - \ln(1000)}{\ln(\sqrt{1.1}) - \ln(0.93)} \approx 19.152
\]

\[
\frac{1}{\log\left( \frac{\sqrt{1.1}}{0.93} \right)} \approx 19.152
\]

(hours)

(Problem continues on the next page.)
The scientist now observes two additional different ant colonies. From her data, it looks like

- Colony C’s population doubles for the first time after 2.5 hours; doubles again 1.5 hours after that; then doubles a third time 1 hour after that.
- Colony D’s population is given by \( P = D(t) = 1200 - 300e^{-0.11t} \), where \( P \) is the number of ants and \( t \) is measured in hours since the experiment started.

e. [2 points] Is Colony C growing exponentially? Circle your answer below. If Yes, find its growth factor. If No, explain why not.

**Yes**  **No**

**Explanation or Growth Factor:**

**Solution:**

No.

Any exponentially growing function should have a constant doubling time. Since the doubling time of this function is changing, it cannot be growing exponentially.

f. [4 points] Find a general formula \( D^{-1}(P) \) and explain what that function means. Show all work.

**Solution:** We need to solve \( P = 1200 - 300e^{-0.11t} \) for \( t \), which will give us \( t \) as a function of \( P \)— in other words, our inverse function.

\[
\begin{align*}
P &= 1200 - 300e^{-0.11t} \\
P - 1200 &= -300e^{-0.11t} \\
\frac{P - 1200}{-300} &= \frac{1200 - P}{300} = e^{-0.11t} \\
\ln\left(\frac{1200 - P}{300}\right) &= -0.11t \\
t &= \frac{1}{-0.11} \ln\left(\frac{1200 - P}{300}\right)
\end{align*}
\]

**Meaning of \( D^{-1}(P) \):**

**Solution:** \( D^{-1}(P) \) gives us the number of hours after the experiment started at which there are \( P \) ants in Colony D.