7. [8 points] Consider the polynomials and \( a(x) = (x + 1)(x^2 - 6x + 3) \) and \( b(x) = x(3x^2 + 2) \).

a. [3 points] Find all the zeros of \( a(x) \) and of \( b(x) \).
(Show your work carefully, and give your answers in exact form.)

**Solution:** To find the zeros of \( a(x) \), we solve \( a(x) = 0 \).

\[
a(x) = 0 \\
(x + 1)(x^2 - 6x + 3) = 0 \\
x + 1 = 0 \text{ or } x^2 - 6x + 3 = 0 \\
x = -1 \text{ or } x = \frac{6 \pm \sqrt{36 - 4(3)}}{2} \\
x = -1 \text{ or } x = 3 \pm \sqrt{6}
\]

Similarly, we solve \( b(x) = 0 \).

\[
b(x) = 0 \\
x(3x^2 + 2) = 0 \\
x = 0 \text{ or } 3x^2 + 2 = 0
\]

\( 3x^2 + 2 = 0 \) has no real solutions (since \( 3x^2 = -2 \) has no real solutions), so the only zero of \( b(x) \) is \( x = 0 \).

**zero(s) of** \( a(x) \): \( x = -1, x = 3 + \sqrt{6}, x = 3 - \sqrt{6} \)  
**zero(s) of** \( b(x) \): \( x = 0 \)

b. [5 points] Let \( r(x) = \frac{a(x)}{b(x)} \).

Find all intercepts and all horizontal and vertical asymptotes of the graph of \( y = r(x) \).

If appropriate, write “NONE” in the answer blank provided.

**Solution:** Since \( a(x) \) and \( b(x) \) have no common zeros, the \( x \)-intercept(s) of the graph of \( y = r(x) \) are the zeros of \( a(x) \) and the vertical asymptote of the graph of \( y = r(x) \) is given by the zero of \( b(x) \).

The \( y \)-intercept of the graph of \( y = r(x) \) is \( r(0) = \frac{a(0)}{b(0)} \) which is undefined since \( b(0) = 0 \).

To find the horizontal asymptote of \( r(x) \), recall that in the long-run, the polynomials \( a(x) \) and \( b(x) \) behave like their leading terms, which are \( x^3 \) and \( 3x^3 \), respectively. Hence, in the long-run, \( r(x) \) behaves like \( \frac{x^3}{3x^3} = \frac{1}{3} \) so \( \lim_{x \to \infty} = \frac{1}{3} \) and \( \lim_{x \to -\infty} = \frac{1}{3} \). Hence the horizontal asymptote of the graph of \( y = r(x) \) is \( y = \frac{1}{3} \).

**\( x \)-intercept(s):** \( -1, 3 + \sqrt{6}, \text{ and } 3 - \sqrt{6} \)

**\( y \)-intercept(s):** None

**horizontal asymptote(s):** \( y = \frac{1}{3} \)

**vertical asymptote(s):** \( x = 0 \)