7. [8 points] Consider the polynomials and $a(x) = (x+1)(x^2 - 6x + 3)$ and $b(x) = x(3x^2 + 2)$.

a. [3 points] Find all the zeros of $a(x)$ and of $b(x)$.
(Show your work carefully, and give your answers in *exact form*.)

Solution: To find the zeros of $a(x)$, we solve $a(x) = 0$.

$$\begin{aligned} a(x) &= 0 \\ (x+1)(x^2 - 6x + 3) &= 0 \\ x + 1 = 0 \text{ or } x^2 - 6x + 3 &= 0 \\ x = -1 \text{ or } x &= \frac{6 \pm \sqrt{36 - 4(3)}}{2} \\ x = -1 \text{ or } x &= 3 \pm \sqrt{6} \end{aligned}$$

Similarly, we solve $b(x) = 0$.

$$\begin{aligned} b(x) &= 0 \\ x(3x^2 + 2) &= 0 \\ x = 0 \text{ or } 3x^2 + 2 &= 0 \end{aligned}$$

$3x^2 + 2 = 0$ has no real solutions (since $3x^2 = -2$ has no real solutions), so the only zero of $b(x)$ is $x = 0$.

zero(s) of $a(x)$: $x = -1$, $x = 3 + \sqrt{6}$, $x = 3 - \sqrt{6}$ **zero(s) of $b(x)$:** $x = 0$

b. [5 points] Let $r(x) = \dfrac{a(x)}{b(x)}$.

Find all intercepts and all horizontal and vertical asymptotes of the graph of $y = r(x)$.
If appropriate, write "NONE" in the answer blank provided.

Solution: Since $a(x)$ and $b(x)$ have no common zeros, the x-intercept(s) of the graph of $y = r(x)$ are the zeros of $a(x)$ and the vertical asymptote of the graph of $y = r(x)$ is given by the zero of $b(x)$.
The y-intercept of the graph of $y = r(x)$ is $r(0) = \frac{a(0)}{b(0)}$ which is undefined since $b(0) = 0$.
To find the horizontal asymptote of $r(x)$, recall that in the long-run, the polynomials $a(x)$ and $b(x)$ behave like their leading terms, which are x^3 and $3x^3$, respectively. Hence, in the long-run, $r(x)$ behaves like $\dfrac{x^3}{3x^3} = \dfrac{1}{3}$ so $\lim\limits_{x \to \infty} = \dfrac{1}{3}$ and $\lim\limits_{x \to -\infty} = \dfrac{1}{3}$. Hence the horizontal asymptote of the graph of $y = r(x)$ is $y = \dfrac{1}{3}$.

x-intercept(s): -1, $3 + \sqrt{6}$, **and** $3 - \sqrt{6}$

y-intercept(s): **None**

horizontal asymptote(s): $y = \dfrac{1}{3}$

vertical asymptote(s): $x = 0$