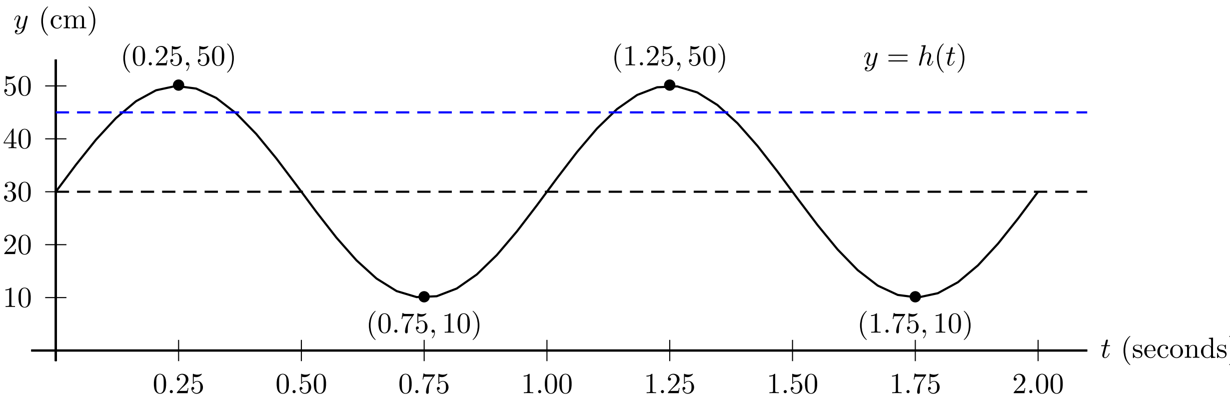
8. [12 points] Suppose Cato is riding a stationary exercise bicycle. His foot moves a pedal in a circle. Let $h(t)$ be the height (in cm) of the pedal above the ground at time t (in seconds). A formula for $h(t)$ is given by

$$h(t) = 20\sin(2\pi t) + 30.$$

a. [3 points] On the axes provided below, graph *two periods* of the function $P = h(t)$ starting with $t = 0$. (*Clearly label the axes and important points on your graph. Be very careful with the* **shape** *and* **key features** *of your graph.*)



b. [2 points] Find the period and amplitude of $P = h(t)$. (*Include units.*)

Period: 1 **second** **Amplitude:** 20 **cm**

c. [4 points] Find all the times t for $0 \le t \le 2$ when the pedal is exactly 45 cm above the ground. (*Find at least one answer algebraically. Show your work carefully and check that your answers make sense.*)

Solution: We are to solve $h(t) = 45$ for $0 \le t \le 2$. The line $y = 45$ has been included in the graph for problem (a). Note that there are 4 points of intersection between $y = 45$ and $h(t) = 45$ for $0 \le t \le 2$. These give us exactly four solutions to the equation $h(t) = 45$ for $0 \le t \le 2$. First we find one solution algebraically:

$$\begin{aligned} h(t) &= 45 \\ 20\sin(2\pi t) + 30 &= 45 \\ 20\sin(2\pi t) &= 15 \\ \sin(2\pi t) &= \frac{3}{4} = 0.75 \end{aligned}$$

One solution is thus given by $2\pi t = \arcsin(0.75)$ which gives $t = \dfrac{\arcsin(0.75)}{2\pi} \approx 0.135$. (This is the smallest of the four solutions.) Note that by symmetry, the next solution is as far to the left of 0.5 as the first solution was to the right of the y-axis. (Alternatively, the average of the first two solutions is 0.25.) Hence the second solution is $0.5 - \dfrac{\arcsin(0.75)}{2\pi} \approx 0.365$. The other two solutions are exactly one period later on the graph, so they are equal to $1 + \dfrac{\arcsin(0.75)}{2\pi} \approx 1.135$ and $1 + \left(0.5 - \dfrac{\arcsin(0.75)}{2\pi}\right) \approx 1.365$.

Answer(s): $t \approx 0.135, 0.365, 1.135,$ **and** 1.365 **seconds**

d. [3 points] Find the length of the arc through which the pedal travels between $t = 0$ and the time the pedal *first* reaches a height of exactly 45 cm.
(*Show your work and reasoning. It may help to sketch a picture.*)

Solution: Since the amplitude of the function $h(t)$ is 20 cm, we see that the radius of the circle around which the pedal is traveling is 20 cm. The pedal is at the 3 o'clock position and ascending at time $t = 0$ and its height above the ground after traveling through an angle of ϕ radians is $20\sin(\phi) + 30$ cm. When it first reaches a height of 45 cm, we have $20\sin(\phi) + 30 = 45$, so $\sin(\phi) = 0.75$. The solution $\phi = \arcsin(7.5)$ corresponds to the first time the pedal reaches 45 cm (as in part (c)). The arc length of the path the pedal had traveled is then $20\arcsin(0.75) \approx 16.96$ cm.

Answer(s): $20\arcsin(0.75) \approx 16.96$ **cm**