5. [5 points] Find a formula for *one* polynomial $p(z)$ that satisfies *all* of the following conditions:

- $\lim_{z\to\infty} p(z) = -\infty$ and $\lim_{z\to-\infty} p(z) = -\infty$
- The only zeros of $p(z)$ are $z = -2$, $z = 1$, and $z = 3$.
- The point $(2, -12)$ is on the graph of $p(z)$.
- The degree of $p(z)$ is at most 5.

Show your work and reasoning carefully. You might find it helpful to first sketch a graph. There may be more than one possible answer, but you should give only one answer.

Solution: Because $\lim_{z\to\infty} p(z) = -\infty$ and $\lim_{z\to-\infty} p(z) = -\infty$, we know that the degree of $p(z)$ is even and the leading coefficient is negative.
Because $p(z)$ has zeros at $z = -2$, $z = 1$, and $z = 3$, the degree of the polynomial must be at least 3. Since it has to be of even degree less than 5, the degree must be 4. Thus there must be exactly one double zero.
Since the point $(2, -12)$ is on the graph, we can see by sketching the graph that the double zero must be at $z = 3$.
So, we have $p(z) = a(z+2)(z-1)(z-3)^2$ for some negative constant a.
Since $p(2) = -12$ we find $-12 = p(2) = a(2+2)(2-1)(2-3)^2 = a(4)(1)(-1)^2 = 4a$ so $a = -3$.
Thus, $p(z) = -3(z+2)(z-1)(z-3)^2$.

Answer: $p(z) = \underline{\quad -3(z+2)(z-1)(z-3)^2 \quad}$

6. [5 points] Find all solutions to the equation

$$5\tan\left(2x + \frac{\pi}{2}\right) - 13 = 12$$

for x between 0 and 5. *Show your work carefully and give your answer(s) in* *<u>exact form</u>.*

Solution:

$$5\tan\left(2x + \frac{\pi}{2}\right) - 13 = 12 \quad \text{so} \quad 5\tan\left(2x + \frac{\pi}{2}\right) = 25 \quad \text{and} \quad \tan\left(2x + \frac{\pi}{2}\right) = 5.$$

Using the inverse tangent function, we find that one solution to the equation is given by

$$2x + \frac{\pi}{2} = \arctan(5) \quad \text{so} \quad 2x = \arctan(5) - \frac{\pi}{2} \quad \text{and} \quad x = \frac{\arctan(5) - \frac{\pi}{2}}{2}.$$

Note that the period of $5\tan\left(2x + \frac{\pi}{2}\right) - 13$ is $\pi/2$ and that the solution $\frac{\arctan(5) - \frac{\pi}{2}}{2}$ is *not* in the interval $[0, 5]$. Other solutions to the equation are obtained by adding integer multiples of the period $\pi/2$ to $\frac{\arctan(5) - \frac{\pi}{2}}{2}$. The resulting solutions in the interval $[0, 5]$ are

$$\frac{\arctan(5) - \frac{\pi}{2}}{2} + \frac{\pi}{2} \quad , \quad \frac{\arctan(5) - \frac{\pi}{2}}{2} + \pi \quad , \text{ and } \quad \frac{\arctan(5) - \frac{\pi}{2}}{2} + \frac{3\pi}{2}.$$

These can be simplified to $\frac{\arctan(5)}{2} + \frac{\pi}{4}$, $\frac{\arctan(5)}{2} + \frac{3\pi}{4}$, and $\frac{\arctan(5)}{2} + \frac{5\pi}{4}$.

Answer: $x = \underline{\quad \frac{\arctan(5)}{2} + \frac{\pi}{4} \quad , \quad \frac{\arctan(5)}{2} + \frac{3\pi}{4} \quad , \textbf{ and } \quad \frac{\arctan(5)}{2} + \frac{5\pi}{4} \quad}$