8. [11 points] On the beaches of Mexico, there is a population of picky snails that wait for special shells to wash up onto the shore. These snails can only live in these particular shells, as the snails have become accustomed to the comfort in these shells.
Suppose the number of hundreds of special shells on the beaches of Mexico t years after the beginning of 2013 is

$$h(t) = (t^2 + 7)(4t - 7)^2$$

and the population, in hundreds, of picky snails t years after the beginning of 2013 is

$$p(t) = (t - 2)^2(8t^2 + 30).$$

Throughout this problem, remember to clearly show your work and reasoning.

a. [3 points] Find the leading term and any zeros of $h(t)$. If appropriate, write "NONE" in the answer blank provided.

Solution: To find the zeros of $h(t)$, we set $h(t) = 0$ and solve for t. Because $t^2 + 7$ is always positive, $h(t) = 0$ when $(4t - 7)^2 = 0$. So the zeros of $h(t)$ occur at $t = 7/4$.
To find the leading term, we take the product of the leading terms of its factors:

$$\text{leading term} = (t^2)(4t)(4t) = 16t^4$$

Answers: Leading Term: $16t^4$ Zero(s): $t = 7/4$

b. [3 points] The number of shells per snail is $Q(t) = \dfrac{h(t)}{p(t)}$.

Find the equations of all vertical asymptotes ("V.A.") and horizontal asymptotes ("H.A.") of the graph of $y = Q(t)$. If appropriate, write "NONE" in the answer blank provided.

Solution: To find vertical asymptotes, since $h(t)$ and $p(t)$ have no common factors, we set $p(t) = 0$ and solve for t. Because $8t^2 + 30$ is always positive, $p(t) = 0$ only when $(t - 2)^2 = 0$. So the only zero of $p(t)$ is $t = 2$. So, since $h(t) \neq 0$, the only vertical asymptote of $Q(t)$ is $t = 2$.
To find horizontal asymptotes, we look at the long-run behavior of $Q(t)$.

$$\lim_{t\to\infty} \frac{h(t)}{p(t)} = \lim_{t\to\infty} \frac{(t^2+7)(4t-7)^2}{(t-2)^2(8t^2+30)} = \lim_{t\to\infty} \frac{16t^4}{8t^4} = 2$$

$$\lim_{t\to-\infty} \frac{h(t)}{p(t)} = \lim_{t\to-\infty} \frac{(t^2+7)(4t-7)^2}{(t-2)^2(8t^2+30)} = \lim_{t\to-\infty} \frac{16t^4}{8t^4} = 2$$

Thus, the horizontal asymptote of $Q(t)$ is $y = 2$.

Answers: V.A.: $t = 2$ H.A.: $y = 2$

There is a competitive population of crabs that live on the same beaches. Suppose that there are 1200 of these crabs at the beginning of 2013, and that the population grows at a *continuous annual rate* of 35%. Let $c(t)$ be the population, in hundreds, of these crabs t years after the beginning of 2013.

c. [2 points] Find a formula for $c(t)$.

Solution: If $c(t) = Pe^{kt}$ is the population, in hundreds, of these crabs t years after the beginning of 2013, then $c(0) = 12$, so $P = 12$. With a continuous annual growth rate of 35%, we know that

$$c(t) = 12e^{.35t}.$$

Answer: $c(t) =$ $12e^{.35t}$

d. [3 points] The crabs like the same special shells as the snails do. Write a formula for the ratio of the number of shells to the number of crabs t years after the beginning of 2013.

Answer: $\dfrac{h(t)}{c(t)} = \dfrac{(t^2+7)(4t-7)^2}{12e^{.35t}}$

In the long run, what happens to the ratio of the number of shells to the number of crabs? In other words, assuming the functions described in this problem continue to be accurate models, what happens to this ratio after many, many years?
You must clearly indicate your reasoning in order to receive any credit for this problem.

Solution: The ratio of number of shells to the number of crabs t years after the beginning of 2013 is given by

$$R(t) = \frac{h(t)}{c(t)}$$

To describe the ratio after many, many years, we look at the long-run behavior. Because we are looking at years after 2013, we only look at the limit of $R(t)$ as t grows without bound, i.e. as $t \to \infty$. Since exponential growth eventually dominates polynomial growth, the limit is zero. That is

$$\lim_{t\to\infty} R(t) = \lim_{t\to\infty} \frac{h(t)}{c(t)} = \lim_{t\to\infty} \frac{(t^2+7)(4t-7)^2}{12e^{.35t}} = \lim_{t\to\infty} \frac{16t^4}{12e^{.35t}} = 0$$

This means that in the long run, the ratio of the number of shells to the number of crabs approaches zero.