5. [11 points] The water levels in a large bay fluctuate due to the tides. High tide, which is when water levels are at their maximum, happens roughly twice per day. Similarly, low tide is when water levels are at their minimum.

a. [6 points] At one particular location in this bay, the depth, in feet, of the water t hours after midnight on December 1 was given by

$$D(t) = 11\cos\left(\frac{24\pi}{149}(t-3)\right) + 56.$$

i. What is the depth of the water, in feet, at this location at high tide? At low tide?

$56 + 11 = 67$ feet at high tide

$56 - 11 = 45$ feet at low tide

ii. Find the period of $D(t)$, either in exact form or rounded to two decimal places. Then interpret what it means in the context of this problem.

Period: $149/12 \approx 12.42$

Meaning:
This is the time, in hours, between high tides. (Equivalently, the time, in hours, between low tides.)

iii. Find the times t of all high tides that occur on December 1. *Give your answer as a list of t-values in exact form or rounded to two decimal places.*

Solution: The graph of the cosine function achieves its maximum value at $t = 0$ and then every period to the left and right of that. Since this is a cosine graph that's been shifted right by 3, it will achieve its maximum at $t = 3$ plus every period to the left and right of that.

$t =$ $3, 3 + 149/12 \approx 15.42$

This problem continues on the next page.

b. [5 points] At another location in the bay, the depth, in feet, of the water t hours after midnight on December 1 was given by

$$P(t) = 9\sin\left(\frac{24\pi}{149}t\right) + 40.$$

Find the t-values of all times on December 1 that the water level at this location was 45 feet. *Give your answer as a list of t-values in exact form or rounded to two decimal places.*

Solution: We can start by fining one solution, then using symmetry to find additional solutions. Algebraically, we can find our first solution as follows:

$$\begin{aligned} 45 &= 9\sin\left(\frac{24\pi}{149}t\right) + 40 \\ 5 &= 9\sin\left(\frac{24\pi}{149}t\right) \\ \frac{5}{9} &= \sin\left(\frac{24\pi}{149}t\right) \\ \arcsin\left(\frac{5}{9}\right) &= \frac{24\pi}{149}t \\ \frac{149}{24\pi}\arcsin\left(\frac{5}{9}\right) &= t \end{aligned}$$

We can use a calculator to help us see that this is approximately equivalent to 1.16 hours, meaning that this is one of the solutions we're looking for on December 1, about 1.16 hours after midnight.

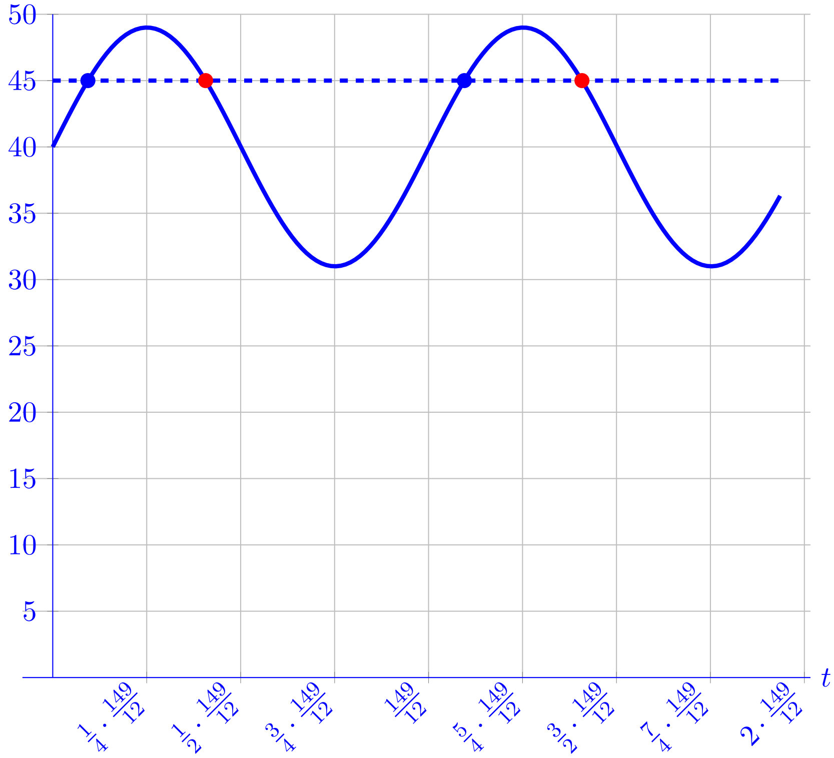
The period of this function is the same as in the earlier part of this problem: $2\pi \div \frac{24\pi}{149} = \frac{149}{12}$. Knowing this, we'll have a second solution one period later at

$$t = \frac{149}{24\pi}\arcsin\left(\frac{5}{9}\right) + 149/12 \approx 13.58$$

However, this is not all of our solutions. The water level passes 45 feet both on its way up to high tide, and again on its way back down to low tide. To find the additional solutions we can use the symmetries of a sin graph (or of a circle). Using the period, midline, amplitude, etc., we can sketch the graph below and see where it intersects the line $y = 45$.

Using a calculator, we can see that the solutions we've found so far correspond to the two blue dots (1st and 3rd dots going left to right). The two we're missing are the red dots below (2nd and 4th dots going left to right).

Solution:



To find the t-values of the red dots, we can notice that the first red dot is as far to the *left* of a half period, as the first blue dot is to the *right* of 0. In other words, the t-value of the first red dot (2nd dot from the left) will be:

$$\frac{1}{2}\cdot\frac{149}{12}-\frac{149}{24\pi}\arcsin\left(\frac{5}{9}\right)\approx 5.044$$

And our final solution will be one period later:

$$\left(\frac{1}{2}\cdot\frac{149}{12}-\frac{149}{24\pi}\arcsin\left(\frac{5}{9}\right)\right)+\frac{149}{12}\approx 17.461$$

$t =$ $\frac{149}{24\pi}\arcsin\left(\frac{5}{9}\right),\ \frac{1}{2}\cdot\frac{149}{12}-\frac{149}{24\pi}\arcsin\left(\frac{5}{9}\right),\ \frac{149}{24\pi}\arcsin\left(\frac{5}{9}\right)+\frac{149}{12}, \frac{1}{2}\cdot\frac{149}{12}-\frac{149}{24\pi}\arcsin\left(\frac{5}{9}\right)+\frac{149}{12}$

Numerically: $t\approx 1.164, 5.004, 13.581, 17.461$