2. [14 points] Psychiatrists and food scientists teamed up to measure how the concentration of different sweeteners affect the perception of sweetness. "Sweetness units" (SU) in this trial ranged from 1 to 15.

- The function $F(C)$ gives the sweetness units (SU) of a C% fructose solution. (A concentration of C% means that, by mass, the fructose was C% of the total solution and the rest was water.)
- The function $A(C)$ gives the sweetness units (SU) of a C% Alitame solution. (A concentration of C% means that, by mass, the Alitame was C% of the total solution and the rest was water.)

a. [6 points] Write an expression or equation, using A, A^{-1}, F, F^{-1}, or their combinations or compositions, that represents each of the following sentences or phrases.

(i) The perceived sweetness of a 6% fructose solution is 9 SUs.

$F(6) = 9$

(ii) The concentration of Alitame that gives a perceived sweetness of 8 SU

$A^{-1}(8)$

(iii) A 0.01% solution of Alitame is 100 times sweeter than a 1% solution of fructose.

$A(0.01) = 100F(1)$

b. [2 points] Which of the following compositions make sense in the context of the problem? *Bubble in all that apply.*

- ◯ $A(F(15))$
- ◯ $F(A(0.05))$
- ◯ $A^{-1}(F(11))$
- ◯ $A(F^{-1}(10))$
- ◯ $A^{-1}(F^{-1}(5))$
- ◯ NONE OF THE ABOVE

Solution: Both $A(F(15))$ and $F(A(0.05))$ try to plug some number of SUs into a function which takes a concentration as input, which doesn't make sense in the context of the problem. Similarly, $A^{-1}(F^{-1}(5))$ tries to plug in a concentration into a function, A^{-1}, which takes as input a number of SUs.

On the other hand, the output of F and the input of A^{-1} are both a number of SUs, so the composition $A^{-1}(F(11))$ does make sense. It is the concentration an Alitame solution would need to be to have the same perceived sweetness (in SUs) as an 11% fructose solution.

The output of F^{-1} is a concentration, and this is what the function A takes as input. Therefore, $A(F^{-1}(10))$ makes sense in the context of the problem. It is the perceived sweetness (in SUs) of an Alitame solution which has the same concentration as a fructose solution which has a perceived sweetness of 10 SUs.

This information is repeated from the previous page for convenience.

- The function $F(C)$ gives the sweetness units (SU) of a $C\%$ fructose solution. (A concentration of $C\%$ means that, by mass, the Fructose was $C\%$ of the total solution and the rest was water.)
- The function $A(C)$ gives the sweetness units (SU) of a $C\%$ Alitame solution. (A concentration of $C\%$ means that, by mass, the Alitame was $C\%$ of the total solution and the rest was water.)

c. [3 points] The scientists found that $F(C)$ was linear, with a slope of 1.33. Given that, which of the following statements about $F(C)$ are TRUE? *Bubble in all that apply.*

◯ A 10% fructose solution will be 33% sweeter than a 9% fructose solution.

◯ An 8% fructose solution will be about 4 SUs sweeter than a 5% fructose solution.

◯ A 4% fructose solution will be 1.33 SUs *less sweet* than a 5% fructose solution.

◯ $F(C)$ is an increasing function.

◯ $F(C)$ has a constant average rate of change.

◯ NONE OF THE ABOVE

Solution: Since $F(C)$ is linear with a positive slope, it is increasing and has a constant average rate of change (of 1.33). If we increase (resp. decrease) the concentration C by 1, the output, which is a perceived sweetness in SUs, increases (resp. decreases) by 1.33. This means that a a 4% fructose solution will be 1.33 SUs less sweet than a 5% fructose solution. Similarly, an 8% fructose solution will be 3.99 SUs, or about 4 SUs, sweeter than a 5% fructose solution. On the other hand, we have that 10% fructose solution will be 1.33 SUs sweeter than a 9% fructose solution, which is not the same as being 33% sweeter.

d. [3 points] The scientists published the following table of data for the function $A(C)$.

C	0.0025	0.005	0.01	0.02
$A(C)$	6.89	9.36	11.4	12.8

Which of the following statements about $A(C)$ may be TRUE, given data in the table provided. *Bubble in all that apply.*

◯ $A(C)$ is increasing.

◯ The average rates of change of $A(C)$ are decreasing.

◯ $A(C)$ is linear.

◯ $A(C)$ is concave up.

◯ $A(C)$ is concave down.

◯ NONE OF THE ABOVE

Solution: We can see from the table that $A(C)$ looks like it could be increasing since the values of $A(C)$ increase as the values of C increase (and this makes sense because a more highly concentrated solution should taste sweeter). The average rates of change of $A(C)$ on the intervals 0.0025 to 0.005, 0.005 to 0.01, and 0.01 to 0.02 are 988, 408, and 140, respectively. Since these are not constant, the function is not linear. Since the average rates of change are decreasing, it could be that $A(C)$ is concave down, but it cannot be concave up.