6. [9 points] Bubble in the blanks for **all possible** correct choices. Use pencil in case you need to change your answer. *You do not need to show work for any part of this problem.*

a. Which of the functions below have the property that: $\lim\limits_{x\to\infty} f(x) = \infty$?

○ $f(x) = \dfrac{9}{x^5}$

○ $f(x) = \dfrac{x^9+5}{2^x+x^2}$

[✓ ○ $f(x) = \dfrac{x^2}{e^{-x}}$]

[✓ ○ $f(x) = \dfrac{x^{\frac{1}{3}}}{\ln(x)}$]

○ NONE OF THE ABOVE

Solution: Let's find the limit as $x \to \infty$ of each of the options.

- Since $x^5 \to \infty$ as $x \to \infty$, $\frac{9}{x^5} \to 0$ as $x \to \infty$.
- We have that $\lim_{x\to\infty} \frac{x^9+5}{2^x+x^2} = \lim_{x\to\infty} \frac{x^9}{2^x} = 0$ since 2^x dominates x^9 as $x \to \infty$.
- As $x \to \infty$, $e^{-x} \to 0$ but is always positive. Since $x^2 \to \infty$ and is always positive as $x \to \infty$, we have, $\lim_{x\to\infty} \frac{x^2}{e^{-x}} = +\infty$.
- Both $x^{1/3}$ and $\ln(x)$ go to ∞ as $x \to \infty$. However, $x^{1/3}$ dominates $\ln(x)$, so we still have $\lim_{x\to\infty} \frac{x^{1/3}}{\ln(x)} = \infty$.

b. Which of the following functions have at least one *horizontal* asymptote?

○ $f(x) = \log(5x)$

[✓ ○ $f(x) = \dfrac{5x^5-x^2}{x^5+x^4}$]

[✓ ○ $f(x) = \dfrac{2}{x^3-x-17}$]

[✓ ○ $f(x) = e^{3x}+1$]

○ $f(x) = \dfrac{x^3+x+71}{x^2-81}$

○ NONE OF THE ABOVE

Solution:

- $\log(5x)$ only has a vertical asymptote and no horizontal asymptotes.
- Since the numerator and denominator of $\frac{5x^5-x^2}{x^5+x^4}$ are polynomials of the same degree, the function has a horizontal asymptote of 5 (which is the ratio of the leading coefficients of the numerator and denominator).
- $\lim_{x\to\infty} \frac{2}{x^3-x-17} = 0$, so this function has a horizontal asymptote at $y = 0$.
- $\lim_{x\to-\infty} e^{3x}+1 = 1$, so this function has a horizontal asymptote at $y = 1$.
- $\lim_{x\to\infty} \frac{x^3+x+71}{x^2-81} = \lim_{x\to\infty} \frac{x^3}{x^2} = \lim_{x\to\infty} x = \infty$. Similarly, $\lim_{x\to-\infty} \frac{x^3+x+71}{x^2-81} = -\infty$. Therefore, this function has no horizontal asymptotes.

c. Which of the following functions have at least one *vertical* asymptote?

[✓ ○ $f(x) = \dfrac{1}{x-5}$]

[✓ ○ $f(x) = \ln(x)+5$]

○ $f(x) = \dfrac{x^2(x-1)^2}{(x-1)}$

○ $f(x) = \dfrac{x^2 - 2x + 5}{x^2 + 1}$

○ NONE OF THE ABOVE

Solution:

- $\frac{1}{x-5}$ has a vertical asymptote at $x = 5$.
- $\frac{x^2(x-1)^2}{(x-1)}$ is undefined at $x = 1$. However, since $x = 1$ occurs as a zero with multiplicity 1 in the denominator but multiplicity 2 in the numerator, it is a hole rather than a vertical asymptote. Therefore, this function has no vertical asymptotes.
- $\ln(x) + 5$ has a vertical asymptote at $x = 0$.
- Since $x^2 + 1$ has no (real) zeros, $\frac{x^2-2x+5}{x^2+1}$ is defined for all real numbers and has no vertical asymptotes.

d. In which of the following equations is y directly proportional to x^2?

○ $y = 2x$

○ $y = x^2 - 5$

[✓ ○ $y = 2x^2$]

[✓ ○ $y = \dfrac{\sqrt{7}x^2}{3}$]

○ $y = \dfrac{4}{x^2}$

○ NONE OF THE ABOVE

Solution: We're looking for the options which have the form $y = kx^2$ for some real number k. The only options of this form are $y = 2x^2$ and $y = \frac{\sqrt{7}x^2}{3}$. If $y = \frac{4}{x^2}$, then y is inversely proportional to x^2, not directly proportional.