10. [11 points] An effective cleaning solution can be made by mixing vinegar and water. Starting with 2 liters of a solution that is one-half water and one-half vinegar, v liters of vineger are added to the solution. Let $C = g(v)$ be the concentration of vinegar in the resulting solution.

That is, $g(v) = \dfrac{\text{Total volume of vinegar}}{\text{Total volume of solution}}$ after v liters of vinegar are added.

a. [2 points] Find a formula for $g(v)$.

Solution: Since the initial two liters of solution is one-half water and one-half vinegar, there is initially one liter of vinegar and one liter of water. When v liters of vinegar are added, the resulting solution has $1 + v$ liters of vinegar and still only one liter of water, so the total volume of solution is $2 + v$ liters. Hence a formula for $g(v)$ is $g(v) = \dfrac{1+v}{2+v}$.

Answer: $g(v) = \dfrac{1+v}{2+v}$

b. [2 points] Describe, in the context of this problem, the behavior of $g(v)$ as $v \to \infty$.

Solution: As $v \to \infty$, the output $g(v)$ approaches 1 (since $\frac{1+v}{2+v}$ behaves like $\frac{v}{v}$ as $v \to \infty$). The graph of $y = g(v)$ has a horizontal asymptote of $y = 1$. In the context of this problem, this means that as more vinegar is added to the solution, the concentration of vinegar in the solution gets closer and closer to 100%. (Since no water is removed, the concentration never actually reaches 100%, but it gets arbitrarily close to 100%.)

c. [4 points] Find a formula for $g^{-1}(C)$.

Solution: We must solve for v in the equation $C = \dfrac{1+v}{2+v}$.

$$\begin{aligned} C &= \frac{1+v}{2+v} \\ C(2+v) &= 1+v \\ 2C + Cv &= 1+v \\ Cv - v &= 1 - 2C \\ v(C-1) &= 1-2C \\ v &= \frac{1-2C}{C-1} \end{aligned}$$

Answer: $g^{-1}(C) = \dfrac{1-2C}{C-1}$

d. [3 points] Find and interpret, in the context of this problem, $g^{-1}(0.75)$.

Solution: Using the formula we found in part (c), we have $g^{-1}(0.75) = \dfrac{1-2(0.75)}{0.75-1} = \dfrac{-0.5}{-0.25} = 2$. In context, this means that in order to achieve a concentration of 75% vinegar, a total of 2 liters of vinegar must be added to the original solution.