MATH 115 — FIRST EXAM

DEPARTMENT OF MATHEMATICS
University of Michigan

October 3, 2001

NAME: ______________________________ ID NUMBER: ____________________________

SIGNATURE: _______________________

INSTRUCTOR: _____________________ SECTION NO: ____________________________

1. This exam has 9 pages including this cover. There are 9 questions.
2. Use of books, notes, or scratch paper is **NOT** allowed. You may certainly use your calculator (but not its manual), and a single 3 inch by 5 inch notecard.
3. **Show all of your work!** Partial credit is available for many problems but can only be given if the graders understand your work. Be sure to explain your reasoning carefully. If you are basing your reasoning on a graph, then sketch the graph. Include units in your answers whenever appropriate.
4. One of the skills being tested in this exam is your ability to interpret detailed, precisely worded, directions. Be sure to read the directions carefully and do all that is asked.
5. Stay calm.

PROBLEM	POINTS	SCORE
1	14	
2	6	
3	8	
4	8	
5	8	
6	13	
7	15	
8	13	
9	15	
TOTAL	100	

1. (2 pts each) True or False? Answer "True" only if the statement is always true.

a) Circle **True** or **False:** If $f(x)$ is a second degree polynomial, then $f(f(x))$ is also a second degree polynomial.

b) Circle **True** or **False:** $e^{a+b} = e^a + e^b$

c) Circle **True** or **False:** If $f(x)$ is an exponential function, then $f(x) \to \infty$ as $x \to \infty$.

d) Circle **True** or **False:** $sin(\pi/3) + x^{ln(e)} + 1$ is a linear function.

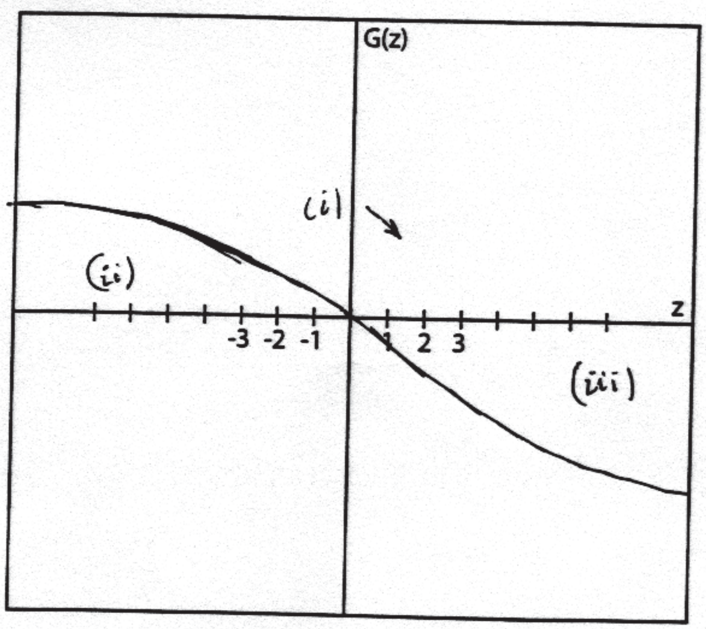
e) Circle **True** or **False:** The derivative of $f(x)$ at a given point is the tangent line at that point.

f) Circle **True** or **False:** If a is positive, then the function $a * ln(x)$ is concave down.

g) Circle **True** or **False:** If $f'(x)$ is an increasing function on an interval, then $f(x)$ is also increasing on that interval.

2. (6 pts) Sketch a graph of a single continuous function $G(z)$ satisfying all of the following conditions:

i) $G'(z)$ is always negative.
ii) When $z < 0$, $G(z)$ is concave down
iii) When $z > 0$, $G(z)$ is concave up.



3. (8 pts) Below is a table of values for the function C(t).

t	3	5	7	9	11	13	15
$C(t)$	2.1	4.2	6.3	8.6	11.5	14.1	18.4

a) Could C(t) be linear? If so, give a formula. If not, demonstrate that it is not. Show your calculations.

No. If $C(t)$ were linear, then $C(t+2) - C(t)$. $C(t)$, for all t. But $C(7) - C(5) = 2.1$ and $C(9) - C(7) = 2.3$, which are not equal.

b) Could C(t) be exponential? If so, give a formula. If not, demonstrate that it is not. Show your calculations.

No. If $C(t)$ were linear, say $C(t) = k \cdot a^t$, then $\frac{C(t+2)}{C(t)}$ would be equal to a^2 for any t. But $\frac{C(5)}{C(3)} = 2$ and $\frac{C(7)}{C(5)} = \frac{3}{2}$.

4. (8 pts) A function $g(x)$ is graphed below, together with its tangent line when $x = 2$.

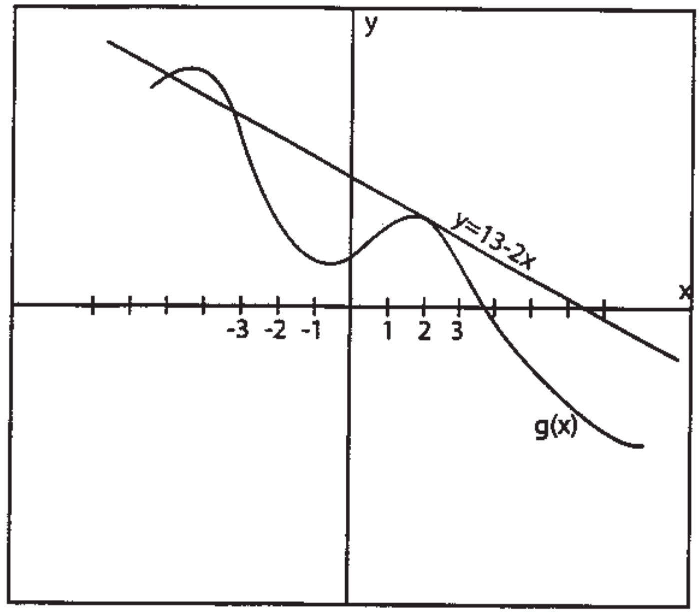


Figure 1: $g(x)$ with tangent line

a) What is the value of $g(2)$?

$g(2) = 9$, since the tangent line crosses the graph of $g(x)$ at the point $(2, g(2))$.

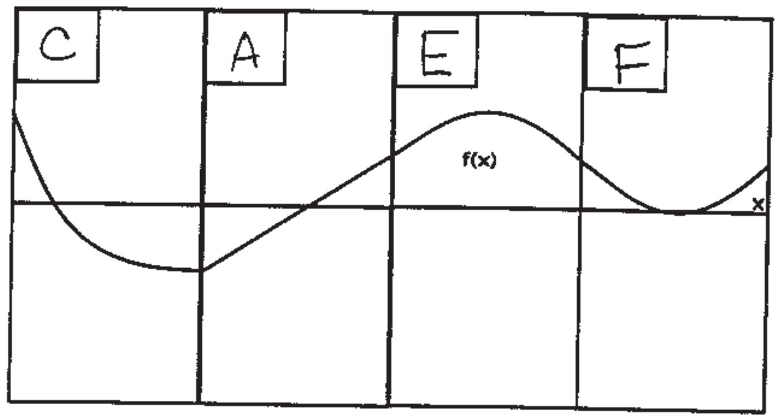
b) What is the value of $g'(2)$?

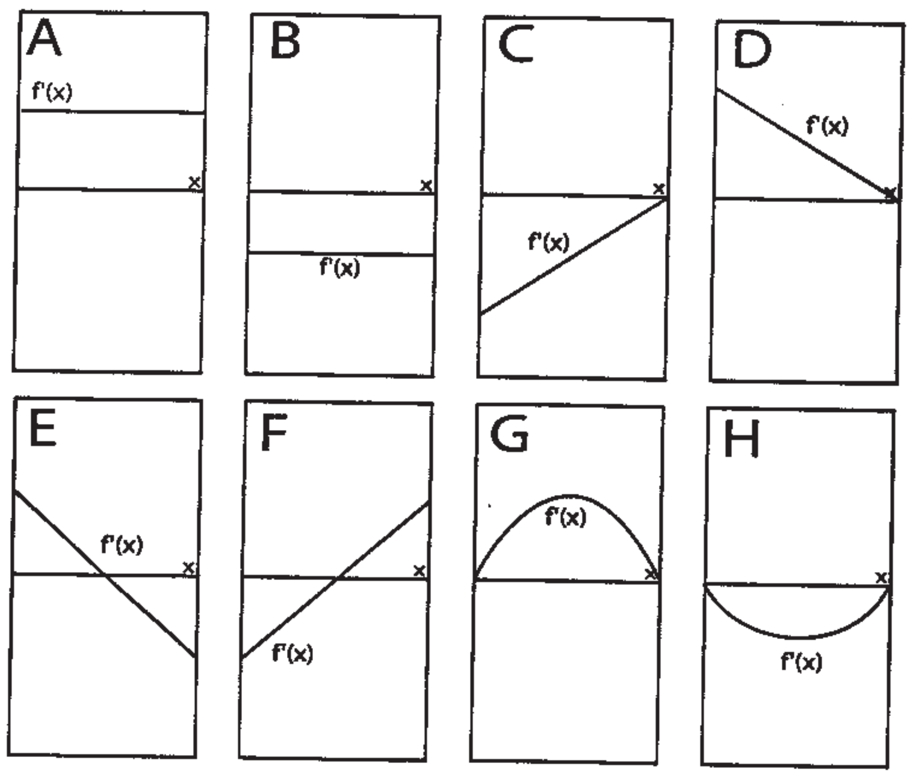
$g'(2) = -2$, since this is the slope of the tangent line to the graph of $g(x)$ at the point $(2, g(2))$.

c) Does the limit $\lim_{h\to 0} \frac{g(2+h)-g(2)}{h}$ exist? If so, what is its value? If not, explain why not.

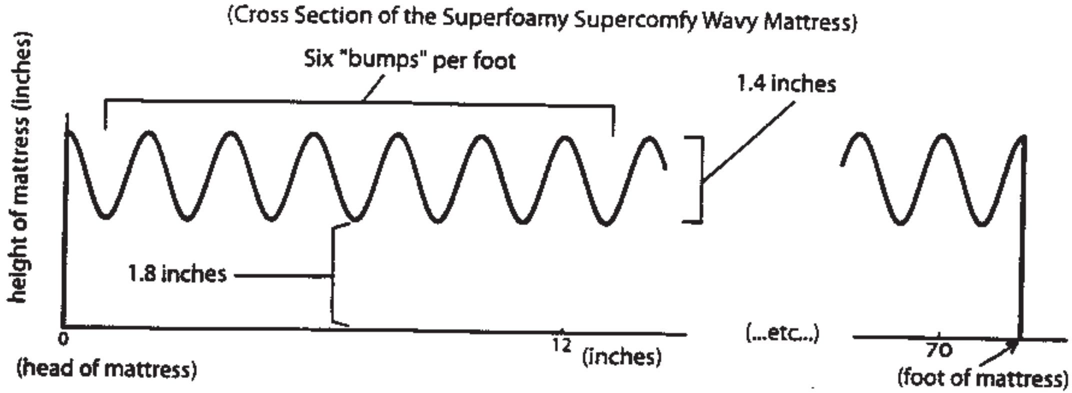
This expression, $\lim\limits_{h\to 0} \frac{g(2+h)-g(2)}{h}$, is the definition using limits of the derivative $g'(x)$ at $x=2$. We are told $g(x)$ has a derivative at $\cancel{x}=2$, so this limit exists. Part (b) says it equals -2 when $x=2$.

5. (8 pts) A function f(x) is graphed below, in four segments. For each segment, choose the graph (A-H) which best describes the derivative of f on that region, and write its letter in the box for that segment. You may use a letter more than once if necessary. (The central horizontal line across each figure A-H is the x-axis.)





6. (13 pts) As part of your new job as Assistant Inventor at "All Things Foam" you have invented a product you call the "Superfoamy Supercomfy Wavy Mattress," and you have delivered the following specifications to Asha, the factory manager:



"I'm sorry," says Asha. "Regulations demand that all new product schematics be given also in formula form."

a) Satisfy the factory regulations by writing a formula for the height H of the mattress, as a function of the horizontal distance d from the head of the mattress.

We will use a cosine function to describe the "wavy profile". We have to adjust the frequency and amplitude to get a correct fit to the picture. We want 36 bumps altogether in 72 inches (or six feet), or one bump every two inches. $\cos\left(\frac{2\pi}{2} d\right)$ will have one peak every 2 units (inches) in d. The amplitude of the wave profile will be $\frac{1}{2} \times 1.4$ in. $= 0.7$ inches, and the wave oscillates around height $1.8 \text{ in} + 0.7 \text{ in} = 2.5 \text{ in}$. Altogether, the profile is described by $h(x) = 2.5 + 0.7 \cos\left(\frac{2\pi d}{2}\right)$

b) What are the domain and range of your function?

The domain is $[0, 72]$ (in units of inches), and the range is $[1.8, 3.2]$.

7. (15 pts) Your local cable internet provider has discovered that the strength (measured in Watts) of the signal generated at their broadcast station decreases fairly rapidly as it travels over their wires. They are concerned about this because subscribers must receive at least a 12-Watt signal in order for their systems to work. Engineers have calculated that $S = f(d) = 160(0.64)^d$, where S is the signal strength, and d represents distance (in miles) from the broadcast station.

a) Translate the statement "$f^{-1}(6) = 11$" into plain English. Is this statement true or false?

The distance from the source where the signal strength is 6 Watts is 11 miles. Not true if $f(d) = 160(0.64)^d$. For $d = 11$, $f(11)$ is much smaller than 6.

b) What percentage of the signal is lost over each mile of cable?

For each gain of 1 in d, the signal strength received is multiplied by 0.64. Thus, a fraction 0.36, or 36 %, is lost each mile.

c) What does the "160" tell you, in real world terms? (answer in a complete sentence.)

The initial signal is 160 Watts strong.

d) How far from the station can one live and still receive the service?

Since $f(d)$ is a decreasing function (exponential function with base $0.64 < 1$), we just solve for d in: $12 = f(d) = 160(0.64)^d$

But then $(0.64)^d = \frac{3}{40}$, $d = \ln(3/40) / \ln(0.64) = 5.804$

e) Better wiring is installed which cuts in half the percentage of the signal lost per mile. You find that the signal strength at your house is doubled! How far from the station do you live?

When this happens, the new function for S is

$g(d) = 160\,(0.82)^d$.

We want to solve $g(d) = 2 f(d)$, or

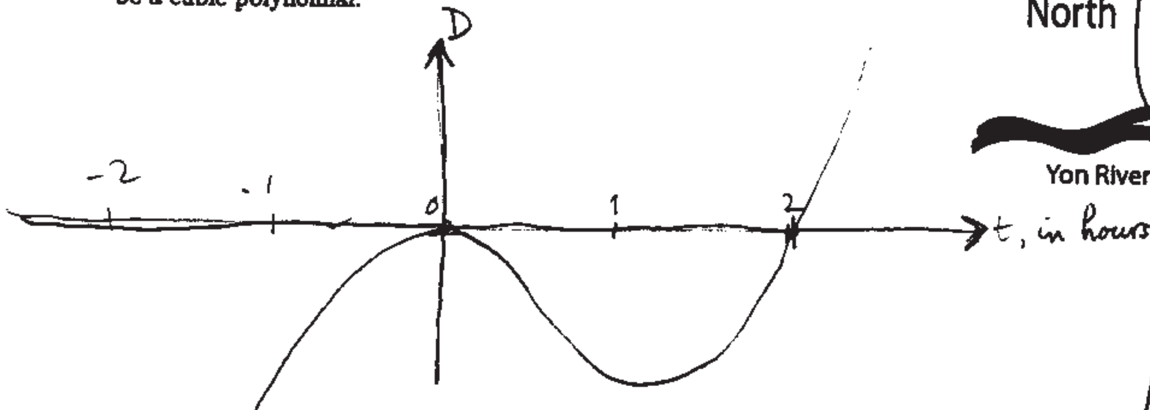
$(0.82)^d = 2\,(0.64)^d$.

Take logs, and solve for d: $d = \dfrac{\ln 2}{\ln(0.82/0.64)} = 2.797.$

8. (13 pts) Johnny Howard, the cubical long-nosed echidna, makes a habit of travelling so that his displacement from Yon River (as a function of time) is always a third degree (cubic) polynomial. This morning he left his home, travelling north to take a basket of scones to his Aunt Hillary. At 12:00 noon, he reached Yon River, but discovered he had forgotten the jam. He then went home again for jam, then back to the river. He crossed the river at 2:00 pm and proceeded to Hillary's house.

Let t be the time in hours after noon (so morning = negative t), and let D be Johnny's displacement north of the river in kilometers (south = negative displacement).

a) Sketch a graph of D against t, keeping in mind that the function must be a cubic polynomial.



b) Write a possible formula for D as a function of t.

$$D(t) = t^2(t-2)$$

This is a cubic polynomial which crosses the t-axis at $t=0$ and $t=2$, and has $D'(0)=0$ as well.

c) Now modify your formula to include some additional information: At 3:00 pm, Johnny was 2 kilometers north of the river.

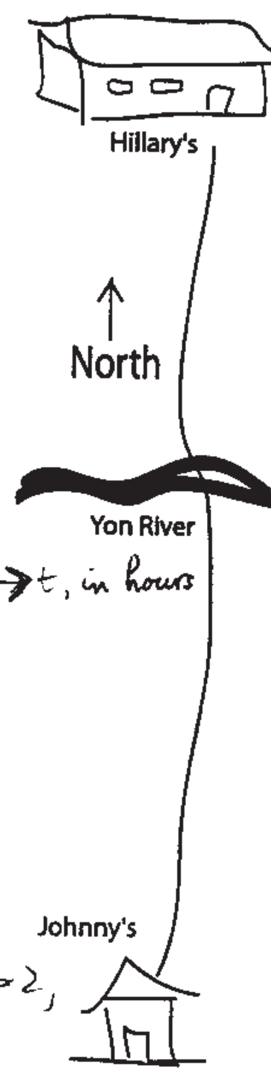


Figure 2: Map of the region

Correct the $D(t)$ in part (b) as follows:

$D(t) = c\,t^2(t-2)$, where c is a positive constant, will also match the graph above.

To fit the (new) condition, $D(3)=2$ requires that $D(3) = c\,3^2(3-2) = 9c = 2$, or $c = \frac{2}{9}$. Thus, $D(t) = \frac{2}{9}t^2(t-2)$ will fit the pattern of the graph and the new condition.

9. (15 pts) The tortoise, the hare, and the rhinoceros begin a 9-mile race at $t = 0$ hours. Remarkably, a 3-way tie results – it takes each competitor exactly 2 hours to finish.

The tortoise's style is slow and steady. He runs the entire race without speeding up or slowing down at all. The hare's style is more erratic: He runs half of the race in the first 20 minutes, stops for a long tea, then runs the second half in the last 20 minutes. The rhino, an amateur mathematician, runs so that her position $R(t)$ in miles from the starting line is always exactly $4.5t^{3-t}$.

a) What is the average velocity on the time-interval $[0, 2]$ of...

i) ...the tortoise?

The average velocity is $\frac{9 \text{ mi}}{2 \text{ hrs}} = 4.5$ mi/hr

ii) ...the hare?

The same average velocity, 4.5 mi/hr

iii) ...the rhinoceros?

The same average velocity, 4.5 mi/hr

b) What is the instantaneous velocity of the tortoise at time $t = 1$?

The tortoise moves with constant velocity, so his instantaneous velocity at $t = 2$ is 4.5 mi/hr.

c) What is the instantaneous velocity of the hare at time $t = 1$?

The hare is stationary from $t = 1/3$ hr. to $t = 1\frac{2}{3}$ hr, so at $t = 1$, its velocity is 0.

d) Estimate the instantaneous velocity of the rhinoceros at time $t = 1$. (Show your work. "I used my calculator" is *not* sufficient work.)

The rhino is at 4.5 mi. at $t = 1$. To estimate the velocity at $t = 1$, compute the average velocity from $t = 1$ to $t = 1.1$. This is $\dfrac{4.5\,(1.1)^{3-1.1} - 4.5\,(1)^2}{0.1}$

≈ 8.934 (By calculator).

e) Imagine that you are a radio reporter describing the events as you see them at time $t = 1$. Tell your audience the status of the race. For example, is anyone passing anyone else?

At $t = 1$, all three are tied at 4.5 mi. The hare is still, the tortoise is moving at speed 4.5 mi/hr and the rhino at speed approximately 8.934 mi/hr. Unless something changes, it looks as though the rhino will win. (Something does change.)