MATH 115 — SECOND EXAM

DEPARTMENT OF MATHEMATICS
University of Michigan

November 7, 2001

NAME: Key ID NUMBER: ______

SIGNATURE: ______

INSTRUCTOR: ______ SECTION NO: ______

1. This exam has 10 pages including this cover. There are 9 questions.
2. Use of books, notes, or scratch paper is **NOT** allowed. You may certainly use your calculator (but not its manual), and a single 3 inch by 5 inch notecard.
3. **Show all of your work!** Partial credit is available for many problems but can only be given if the graders understand your work. Be sure to explain your reasoning carefully. If you are basing your reasoning on a graph, then sketch the graph. Include units in your answers whenever appropriate.
4. One of the skills being tested in this exam is your ability to interpret detailed, precisely worded directions. Be sure to read the directions carefully and do all that is asked.
5. Stay calm.

PROBLEM	POINTS	SCORE
1	6	
2	12	
3	12	
4	6	
5	6	
6	15	
7	12	
8	18	
9	13	
TOTAL	100	

1. (6 points) Let w be the thickness (in inches) of the insulating clothing chosen by a mountain climber. Let $T(w)$ be her resulting body temperature, in Fahrenheit degrees.

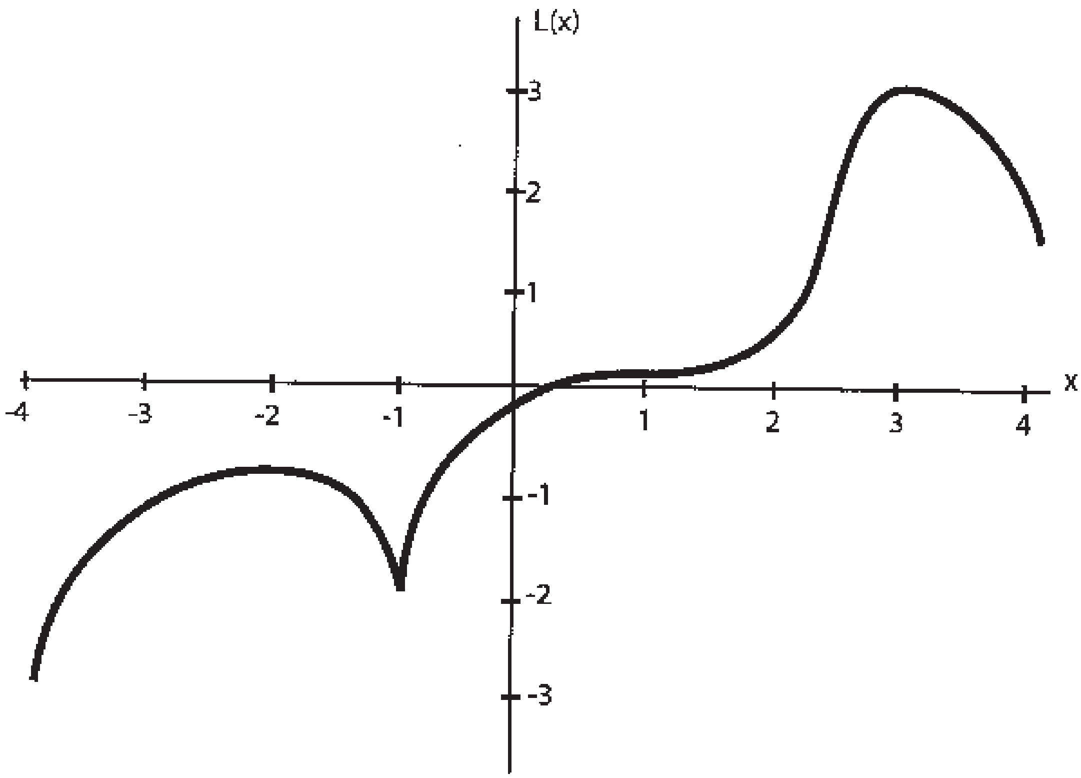
a) (4 pts) What does the formula $T'(4) = -0.4$ tell you, in terms of temperature and clothing?

This means that at 4 inches thickness of clothing, the body temperature is approximately decreasing by $\frac{4}{10}$ °F if you add an extra inch of insulation.

b) (2 pts) Is the statement of part a) reasonable or unreasonable? No explanation necessary.

This doesn't really make sense, since one would expect $T'(w) \geqslant 0$, i.e., the more insulation she wore, the warmer she would stay.

2. (12 points) A function $L(x)$ is graphed below. The domain of L is $[-4, 4]$ (i.e., just the region shown).



a) (4 pts) List all the values of x which are critical points. No explanation necessary.

$x =$ $\boxed{-2, -1, 1, 3}$

b) (2 pts) Which value of x produces the global maximum value of $L(x)$? No explanation necessary.

$x =$ $\boxed{3}$

c) (2 pts) Which value of x produces the global minimum value of $L(x)$? No explanation necessary.

$x =$ $\boxed{-4}$

c) (4 pts) List all the values of x which give points of inflection. No explanation necessary.

$x =$ $\boxed{1, 2.5}$

"values are approximate"

3. (12 points) This table describes two functions, $f(x)$ and $g(x)$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	π	1.4	1.5	-0.2
3	5	1	1	3
5	2.5π	0.3	6	4

a) (4 pts) Find $h'(3)$, assuming $h(x) = f(g(x))$. Show your work.

$$h'(3) = f'(g(3))\cdot g'(3)\text{, by the Chain Rule.}$$
$$= f'(1)\cdot g'(3)$$
$$= (1.4)\cdot(3) = 4.2$$

b) (4 pts) Find $j'(5)$, if $j(x) = \frac{f(x)}{g(x)}$. Show your work.

$$j'(5) = \frac{g(5)\, f'(5) - f(5)\, g'(5)}{g(5)^2} \quad \text{(by the Quotient Rule)}$$
$$= \frac{6\cdot(0.3) - (2.5\pi)\,4}{36} = \frac{1.8 - 10\pi}{36}$$

c) (4 pts) Find $k'(5)$, where $k(x) = xg(x)$. Show your work.

$$k'(x) = 1\cdot g(x) + x\cdot g'(x)\text{, by the Product Rule.}$$

$$k'(5) = 1\cdot 6 + 5\cdot 4 = 26$$

4. (6 points) Below is a graph of $q'(x)$, the derivative of $q(x)$.

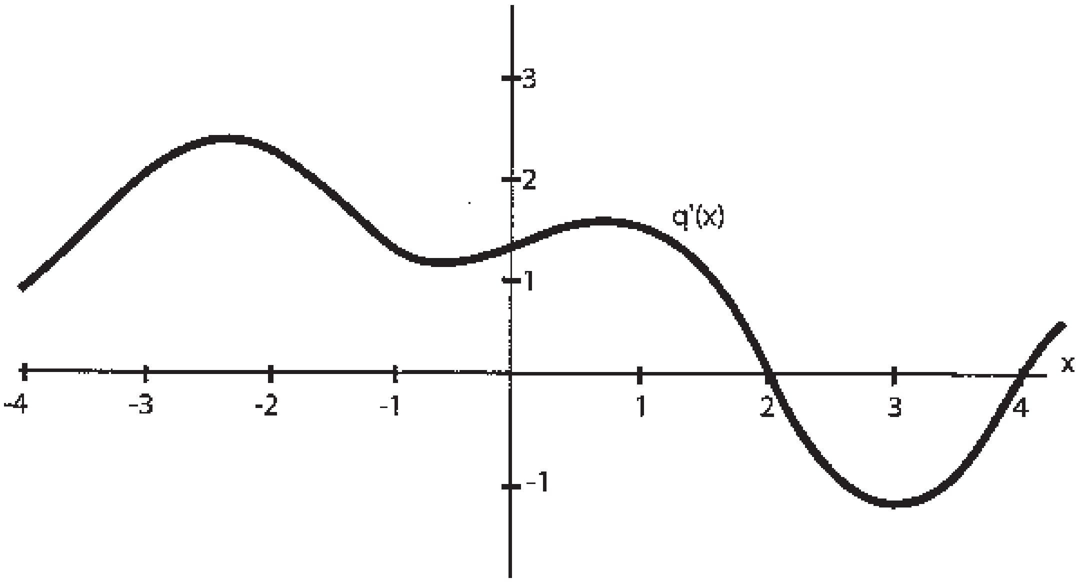


Figure 1: NOT the graph of q(x)

For what values of x is $q(x)$ both decreasing and concave down? Explain.

If q is decreasing, then $q'(x) \leq 0$, so x has to be between 2 and 4.

If q is concave down, then the derivative q' must be decreasing. If it (q') also has to be ≤ 0, then x must be between 2 and 3:

$$2 \leq x \leq 3.$$

5. (6 points) The function $\Phi(x)$ is approximated near $x = 0$ by the 4th degree Taylor polynomial:

$$\Phi(x) \approx x - x^2 + 7x^3 - \frac{\pi}{24}x^4$$

a) (3 pts) Calculate $\Phi'''(0)$.

We can compute this matching the coefficient given for the Taylor polynomial:

coefficient of $x^3 = 7 = \dfrac{\Phi'''(0)}{3!} = \dfrac{\Phi'''(0)}{6}$

So, $\boxed{\Phi'''(0) = 42.}$

b) (3 pts) Is $\Phi(x)$ concave up, concave down, or neither near $x = 0$? Explain *without* using a graph.

Just as above, we can figure:

coefficient of $x^2 = -1 = \dfrac{\Phi''(0)}{2!}$,

or $\Phi''(0) = -2.$

Therefore, Φ is concave <u>down</u> near $x = 0$.

6. (15 points)

a) (6 pts) Find the equation of the tangent line of $f(x) = tan(x)$ at the point $x = \pi/4$.

$$f'(x) = \frac{1}{\cos^2 x} \; ; \; f'(\pi/4) = \frac{1}{(\sqrt{2}/2)^2} = 2$$

Tangent line has equation $y = \tan\left(\frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)$

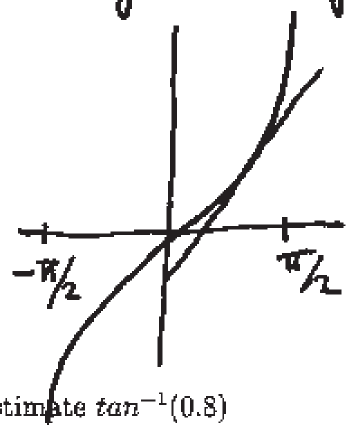
$$y = 1 + 2\left(x - \frac{\pi}{4}\right).$$

b) (2 pts) Use part a) to estimate $tan(3/4)$.

$$\tan\left(\frac{3}{4}\right) \approx 1 + 2\left(\frac{3}{4} - \frac{\pi}{4}\right) = 2.5 - \frac{\pi}{2}.$$

c) (4 pts) Circle **True** or **False**: The actual value of $tan(3/4)$ is greater than your estimation from the previous part. (Briefly comment on the shape of the graph to justify your answer.)

Comment: The graph of $y = \tan x$ looks like this



In particular, it is concave up for $0 < x < \pi/2$. The tangent line lies below the graph for such x.

d) (3 pts) Use part a) to estimate $tan^{-1}(0.8)$

$$x = \tan^{-1}(0.8)$$

implies $\tan x = 0.8$

But $\tan x \approx 1 + 2\left(x - \frac{\pi}{4}\right)$,

so we can solve $0.8 = 1 + 2\left(x - \frac{\pi}{4}\right)$

to get $\boxed{x = \frac{\pi}{4} - 0.1}$ as an estimation of $\tan^{-1}(0.8)$.

7. (12 points) A bungee jumper's height above a river (h in meters) and velocity (v in meters per second — positive v is upward motion) are related. ("Bungee Jumping" is the sport of jumping usually head-first from a tall bridge while securely fastened by an elastic cord. A bungee jumper will bob up and down for some time after being caught by the cord.)

The algebraic relationship between v and h turns out to be:

$$5v^2 + h^2 - 102h = 500$$

a) (3 pts) The jumper later exclaims: "Dude, I was like 36 meters above the river and bouncing up at like 24 meters per second! Rock on!" but his mother suspects he was exaggerating. Demonstrate that his claim is indeed approximately correct.

We want to know v when h is 36: plug $h=36$ above to get $v^2 = 575.2$. If $v=24$, $(24)^2 = 576$; since v must be >0, $v \approx 24$.

b) (6 pts) Using implicit differentiation, calculate $\frac{dv}{dh}$ in terms of v and h.

Differentiate $5v^2 + h^2 - 102h = 500$

to get

$$10v\frac{dv}{dh} + 2h - 102 = 0$$

or $$\frac{dv}{dh} = \frac{-2(h-51)}{10v} = \frac{51-h}{5v}$$

c) (3 pts) Calculate $\frac{dv}{dh}$ at the moment described by the jumper in part a).

$$\frac{dv}{dh} = \frac{51-36}{5\cdot\sqrt{575.2}} = \frac{3}{\sqrt{575.2}}$$

8. (18 points) (For full credit, show all your work on each part of this question.) Mushroom growth $G = f(x)$ in a controlled environment is a function of light intensity x. Specifically,

$$G = f(x) = (x^2 + 2x + 2 - Q) * e^{-x},$$

where Q is a constant depending on the species of mushroom. We did some differentiation for you:

$$f'(x) = (-x^2 + Q) * e^{-x}$$
$$f''(x) = (x^2 - 2x - Q) * e^{-x}$$
$$f'''(x) = (-x^2 + 4x - 2 + Q) * e^{-x}$$

a) (5 pts) Taking Q to be an unknown constant, find the values of all critical points of this function, assuming its domain is $(-\infty, \infty)$.

Set $f'(x) = 0 = (-x^2 + Q)\, e^{-x}$.
This implies $0 = -x^2 + Q$, or $x^2 = Q$.
If $Q < 0$, there are <u>no</u> critical points. If $Q \geq 0$, $x = \pm\sqrt{Q}$

b) (2 pts) The Greater Mycoparadeigma mushroom has $Q = .81$. What are the critical points of its growth function? Again assume its domain is $(-\infty, \infty)$.

$x = \pm\sqrt{.81} = \pm 0.9$

c) (6 pts) Is each critical point found above a local minimum, a local maximum, or neither? (Use any method, but indicate how you know.)

Check the second derivative f'' at ± 0.9:

$f''(0.9) = (0.81 - 2 \cdot 0.9 - 0.81)\, e^{-0.9} < 0$,

$f''(-0.9) = (0.81 + 2 \cdot 0.9 - 0.81)\, e^{+0.9} > 0$.

So, $x = 0.9$ is a local max, $x = -0.9$ local min.

d) (5 pts) Your variable-intensity bulb can be set at x=0, x=4, or anywhere in between. What is the optimal lighting intensity for the Greater Mycoparadeigma? Show your work so we know you have been thorough!

The local max for $0 < x < 4$ is at $x = 0.9$ by part (c). We still need to compare this to the endpoints: $f(0) = 2 - 0.81$

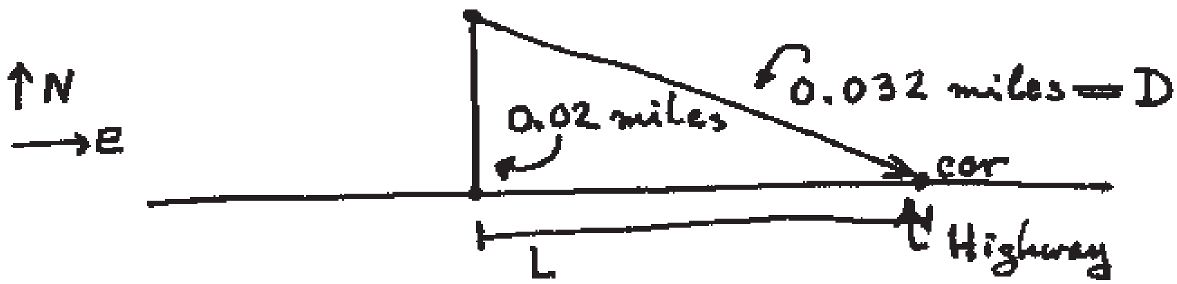
$f(4) = (22 - 0.81) \cdot e^{-4} \leq \frac{22}{16} \approx 1.38$. But $f(0.9) \approx 1.54$

Or, you can observe that $f'(x) < 0$ for all $x > 0.9$, so f decreases from 0.9 to 4.

9. (13 points) A car is driving east along a straight highway, and Police Officer Doright is stationed 0.02 miles north of the highway. After the car passes by, Officer Doright attempts to measure its position and velocity with his radar detector. This device tells him the distance D between himself and the car. Instead of telling him the true velocity of the car, it tells the instantaneous rate of change of the distance between him and the car.

The radar detector reads "D: 0.032 miles, dD/dt: 65 mph".

a) (5 pts) Draw a diagram indicating the positions of the road, of Officer Doright and of the other car. Include all information you can, and label your picture clearly.



a) (8 pts) Calculate the velocity of the car.

The car's velocity is just $\frac{dL}{dt}$, where L is the distance marked along the road.

But $D(t)$ is the length of the hypotenuse and so $L(t) = \sqrt{D^2(t) - (0.02)^2}$.

Thus, $$\frac{dL}{dt} = \frac{D(t)\,D'(t)}{\sqrt{D^2(t) - (0.02)^2}}.$$

We know, then, $$\frac{dL}{dt} = \frac{(0.032)\cdot 65}{\sqrt{(0.032)^2 - (0.02)^2}} \approx 83.27$$

(Units are MPH.)