MATH 115 — FINAL EXAM

DEPARTMENT OF MATHEMATICS
University of Michigan

December 18, 2001

NAME: Key ID NUMBER: ______________

SIGNATURE: ______________

INSTRUCTOR: ______________ SECTION NO: ______________

1. This exam has 12 pages including this cover. There are 13 questions.
2. Use of books, notes, or scratch paper is **NOT** allowed. You may certainly use your calculator (but not its manual), and a single 3 inch by 5 inch notecard.
3. **Show all of your work!** Partial credit is available for many problems but can only be given if the graders understand your work. Be sure to explain your reasoning carefully. If you are basing your reasoning on a graph, then sketch the graph. Include units in your answers whenever appropriate.
4. One of the skills being tested in this exam is your ability to interpret detailed, precisely worded directions. Be sure to read the directions carefully and do all that is asked.
5. Stay calm.

PROBLEM	POINTS	SCORE
1	7	
2	5	
3	5	
4	5	
5	8	
6	8	
7	6	
8	7	
9	9	
10	10	
11	8	
12	12	
13	10	
TOTAL	100	

1. (7 pts) Let $q(x) = (1 + e^{-x})^{-1}$

a) (4 pts) Calculate the derivative $q'(x)$. No need to simplify.

$$q'(x) = \frac{-(1+e^{-x})'}{(1+e^{-x})^2} \quad \text{(power law)}$$

$$= \frac{-(-e^{-x})}{(1+e^{-x})^2} \quad \text{(exponential/chain rule)}$$

$$= e^{-x}/(1+e^{-x})^2$$

b) (3 pts) Imagine inserting your answer from part a) into the integral below. What does the fundamental theorem of calculus tell you about the integral? You need not calculate a value.

$$\int_{-1000}^{1000} (\text{answer from part a})\,dx = \int_{-1000}^{1000} q'(x)\,dx$$

$$= q(1000) - q(-1000)$$

or, $= (1 + e^{-1000})^{-1} - (1 + e^{1000})^{-1}$

2. (5 pts) $\int_3^9 (5 - 2f(x))dx = 22$. Find $\int_3^9 f(x)dx$.

$$22 = \int_3^9 (5 - 2f(x))dx = \int_3^9 5\,dx - 2\int_3^9 f(x)\,dx$$

$$= 30 - 2\int_3^9 f(x)\,dx$$

solving, $2\int_3^9 f(x)\,dx = 30 - 22 = 8$

$$\int_3^9 f(x)\,dx = 4$$

3. (5 pts) Use the Fundamental Theorem of Calculus to calculate the exact value of the integral $\int_3^4 (\frac{1}{x} - 4x)dx$. **Show your work.**

If $f'(x) = \frac{1}{x} - 4x$, then

$$f(x) = \ln x - 4\frac{x^2}{2} + C$$
$$= \ln x - 2x^2 + C$$

(we won't need C to evaluate a definite integral)

So, $\int_3^4 (\frac{1}{x} - 4x)\,dx = \ln x - 2x^2 \Big|_3^4 = \ln 4 - 32 - \ln 3 + 18 = \ln\frac{4}{3} - 14$

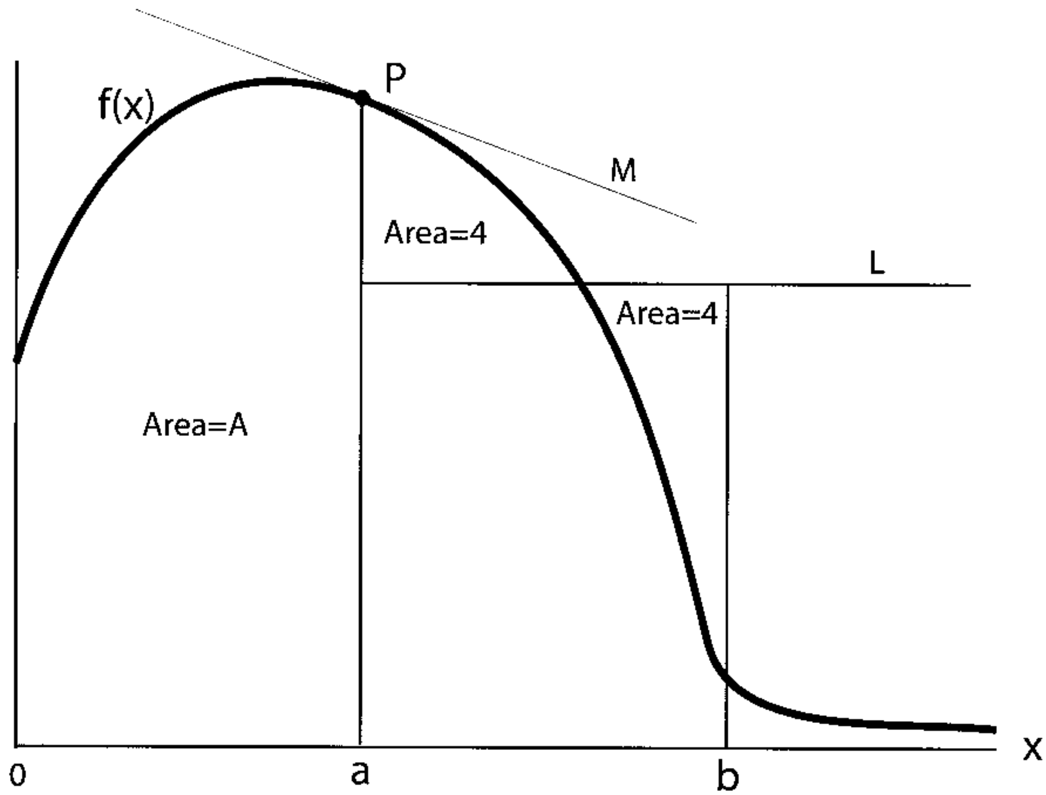
4. (5 pts) Given that $g'(3) = 8$, and that when $x = 3$, $\frac{d}{dx}f(g(x)) = 20$, find $f'(g(3))$.

$$\frac{d}{dx} f(g(x)) = f'(g(3))\,g'(3) \quad \text{(chain rule)}$$

Substituting the given values:

$$20 = f'(g(3)) \cdot 8$$
$$f'(g(3)) = \frac{5}{2} = 2.5$$

5. (8 pts) Below you will write expressions for each of various quantities indicated on the graph of $f(x)$. Your expressions may involve integrals or derivatives. For example, if asked for the "x-coordinate of the point P," you would write "a".



a) (2 pts) The height (above the x-axis) of the point P.

This is the value of $f(x)$ at $x = a$, that is, $f(a)$.

b) (2 pts) The slope of the line M

$f'(a)$

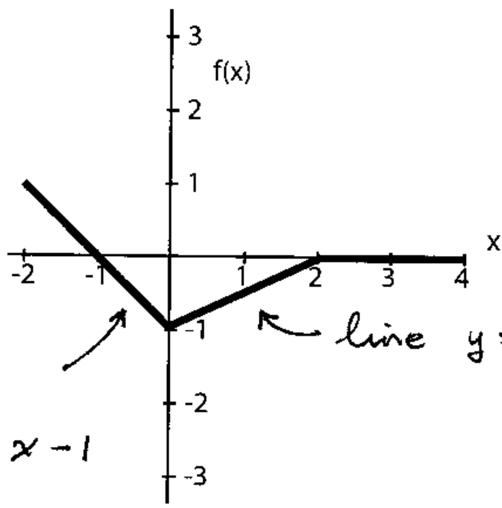
c) (2 pts) The size of the area A.

$$A = \int_0^a f(x)\,dx$$

d) (2 pts) The height of the line L

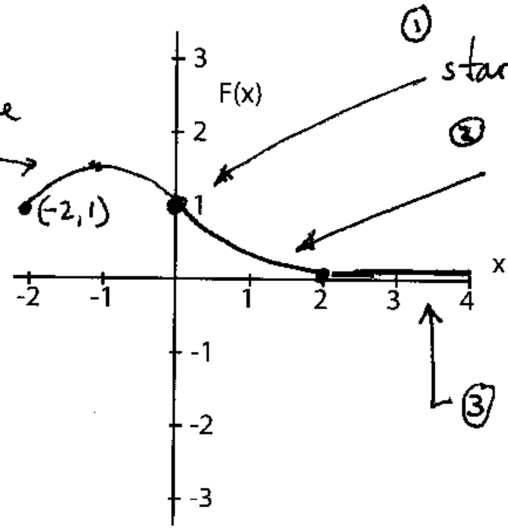
This is the average value of $f(x)$ over the interval $[a, b]$, that is, $\frac{1}{b-a}\int_a^b f(x)\,dx$.

6. (8 pts) The function $f(x)$ is graphed below. Accurately sketch the antiderivative $F(x)$ of $f(x)$. Assume that $F(0) = 1$.



line $y = \frac{1}{2}x - 1$

line $y = -x - 1$



① start from here!

② a parabola, concave up, min at $x = 2$, equation $\frac{x^2}{4} - x + 1$.

③ constantly 0 for $x \geq 0$.

④ parabola, concave down, max at $x=-1$, symmetric from $x=0$ to $x=2$, equation $\frac{-x^2}{2} - x + 1$

7. (6 pts) In this problem we will investigate the family of functions

$$f(x) = a\ln(x) - bx.$$

Calculate values of a and b which cause such a function to have a critical point at $(2, 1)$.

For the graph of $f(x)$ to pass through the point $(2,1)$ we need:

$$① \quad a\ln(2) - b \cdot 2 = 1$$

For $(2,1)$ to be a critical point, we need $f'(2) = 0$, that is; since

$$f'(x) = \frac{a}{x} - b, \text{ we want}$$

$$② \quad f'(2) = 0 = \frac{a}{2} - b.$$

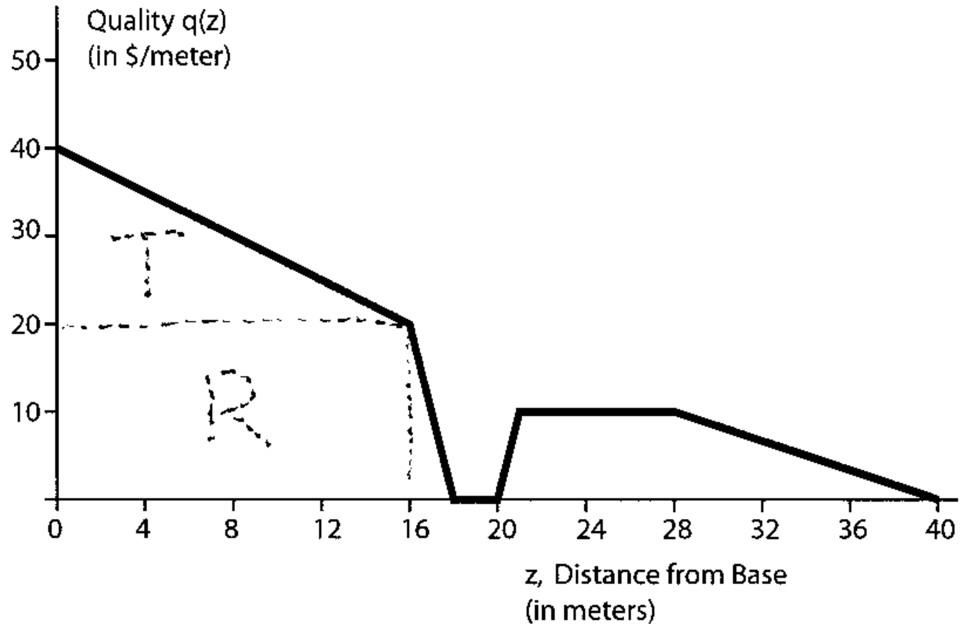
From ②, we get $b = \frac{a}{2}$. Substitute this into ① to get

$$1 = a\ln 2 - \frac{a}{2} \cdot 2 = a\ln(2) - a$$

$$= a(\ln(2) - 1)$$

Solving, $a = \dfrac{1}{\ln(2) - 1}$, $b = \dfrac{1}{2(\ln(2) - 1)}$.

8. (7 pts) A particular treetrunk 40 meters long will be cut for lumber. Let z be the distance from the tree's base to another point on the tree. At each point z, the lumber company assesses the quality $q(z)$ (in *dollars per meter*) of the tree at that point. This measurement — worth per meter's length — is higher at the base of the tree (because it is wider there), and can vary due to "imperfections" in the wood.



a) (4 pts) Calculate or estimate $\int_0^{16} q(z)dz$. **Show your work.**

Integrate "visually": the area under the graph between $z=0$ and $z=16$ = sum of area of the rectangle R plus the area of triangle T = 320 + 160
= 480 (Units are $.)

b) (3 pts) One of the following conclusions from your work in part a is correct (when the answer from a is plugged into the blank space), and the rest are incorrect. Find the correct conclusion, circle it, and fill in the blank with your answer from part a, including units.

Which is true?

~~I.~~ After 16 seconds of processing, the value of the tree goes up by ________.

~~II.~~ The overall change in the value of the whole tree is ________.

III. The value of the first 16 meters of the tree is $480.

~~IV.~~ A tree of height 16 meters is worth ________ more than a tree of height 0 meters.

~~V.~~ The surface area of a tree 16 meters tall is ________.

9. (9 pts) Below is a table of values for the velocity of a downhill skier.

t (seconds)	0	1.5	3	4.5	6	7.5	9	10.5
$v(t)$ (meters/second)	4	19	25	28	31	33	33	32

Calculate or estimate each of the quantities below. **Include units.**

a) (2 pts) The skier's instantaneous velocity at $t = 6$.

$v(6) = 31$ meters/second

b) (3 pts) The distance traveled by the skier in the first 4.5 seconds.

$1.5 \times 4 + 1.5 \times 19 + 1.5 \times 25$

$\approx 6 + 28.5 + 37.5 = 72$ meters.

c) (2 pts) The skier's instantaneous acceleration at time $t = 4.5$.

$$a(4.5) \approx \frac{v(6) - v(4.5)}{1.5} = \frac{31 - 28}{1.5} = 2.$$

(units = meters/sec^2)

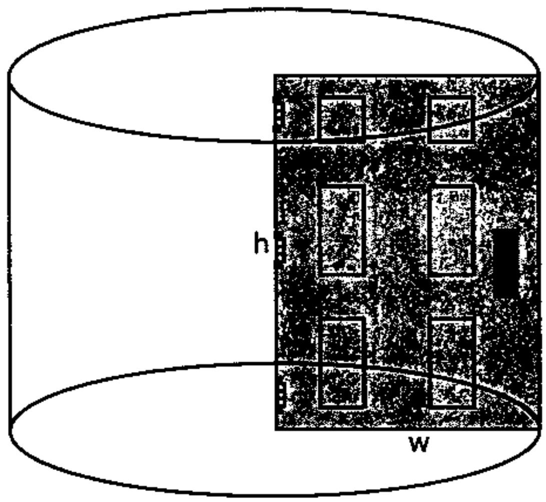
d) (2 pts) The skier's average acceleration over the first 4.5 seconds.

$$\text{Average acceleration} = \frac{v(4.5) - v(0)}{4.5}$$

$$= \frac{28 - 4}{4.5}$$

$$= \frac{24}{4.5} = 5.33 \text{ meters/sec}^2.$$

10. (10 pts) Your "Superfoamy Supercomfy Mattress" was a big hit! You've been promoted to Manager of the *Impractical Doors* subdivision of All Things Foam. Your latest brainchild is "The Great Transcen-Door," a door which rotates 360 degrees around its hinge. (See diagram). Architectural constraints force the volume of the cylinder in which the door rotates to be 3000 cubic feet. You believe that the door will be most pleasing if it has small perimeter. How can you design the (rectangular) door so that the perimeter of this rectangle is as small as possible? What is that perimeter? (The volume of a cylinder is $\pi r^2 h$, where r is the base radius and h the height. The door may indeed be impractical.)



We are given $\pi h w^2 = 3000$ cu. ft.

The perimeter of the door is $2h + 2w$.

Write $h = \dfrac{3000}{\pi w^2}$ (from the volume eqn.),

and substitute into the formula for the perimeter to get $P(w) = 2w + \dfrac{6000}{\pi w^2}$.

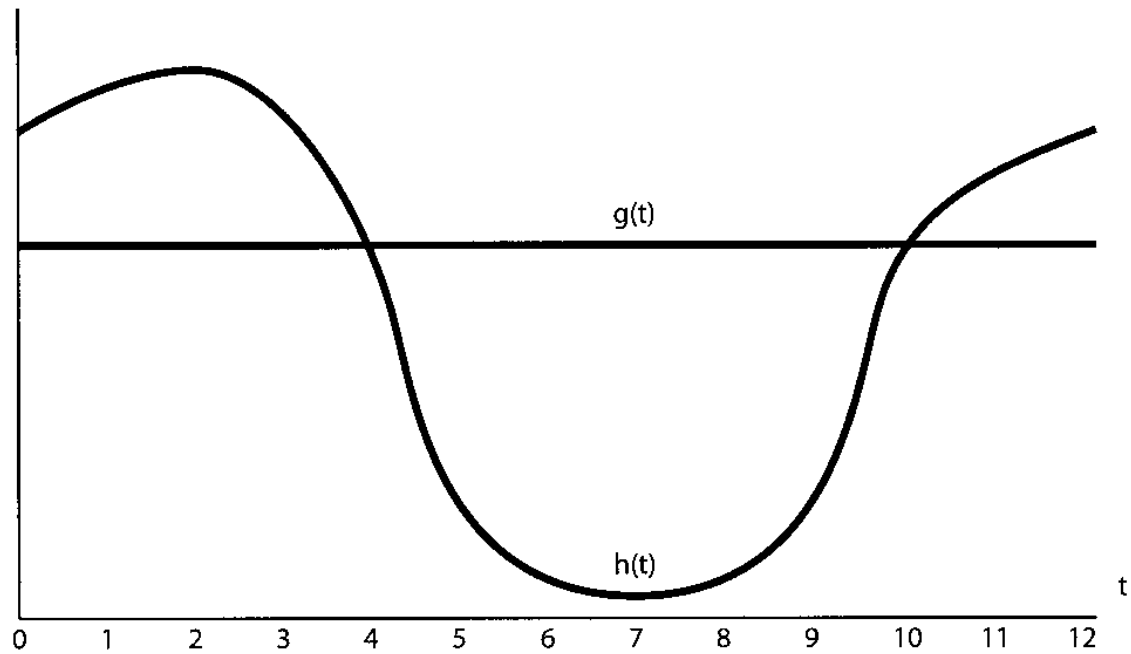
To minimize P, set $P' = 0 = 2 - \dfrac{2 \cdot 6000}{\pi w^3}$.

Solving gives $\pi w^3 = 6000$, or

$w = \sqrt[3]{\dfrac{6000}{\pi}}$, $h = \sqrt[3]{\dfrac{3000}{4\pi}}$. (Units = feet)

Or, $w = 12.4$ ft., $h = 6.2$ ft., $P = 37.2$ ft.

11. (8 pts) Many miles north of here, Yon Glacier is fed throughout the year by new snowfall. New snow arrives at a constant, steady rate. Snow melts off of the glacier at a rate which varies with the season. Below is a graph of the rates of snowfall and snowmelt, both functions of time t in months, where $t = 0$ corresponds to some unknown month (not necessarily January).



a) (2 pts) Is $g(t)$ the rate of snowfall or the rate of snowmelt?

It is the rate of snowfall (constant).

b) (3 pts) What value of t is probably January?

January is probably $t = 6$ or 7, when snow melt is least (approximately). NB: glacier is in the Northern Hemisphere.

c) (3 pts) At what value of t is Yon Glacier the smallest?

It should be ~~greater~~ smallest at $t = 4$; from $t = 4$ until $t = 10$, it is growing, since $h(t) < g(t)$ there. And the area between $h(t)$ ~~[illegible]~~ and $g(t)$ from $t = 4$ to $t = 10$ is greater than the area between them from $t = 10$ to $t = 12$, so, although the glacier shrinks ~~[illegible]~~ from $t = 10$ to $t = 12$, it doesn't ~~[illegible]~~ lose all that it ~~lost~~ gained from $t = 4$ to $t = 10$.

12. (12 pts) Suppose that a company (called All Things Food) has hired you as a consultant. You are to help them save their failing product, "Big J's Bar-B-Q Ice Cream." You have discovered that their cost and revenue functions (in dollars) are:

$$\mathbf{C(q) = 100 + 2q} \qquad \text{and} \qquad \mathbf{R(q) = 15q^{.75}},$$

where q is the number of ice cream containers produced.

a) (1 pt) What is the product's fixed cost?

Fixed cost $= 100$ (dollars).

b) (3 pts) Last year, All Things Food produced 2400 containers of Big J's Bar-B-Q Ice Cream. What was their profit?

Profit $= R(2400) - C(2400) = 15(2400)^{3/4} - 4900$

c) (5 pts) Find formulas for marginal cost and marginal revenue, and evaluate at $q = 2400$.

$MC(q) = (100 + 2q)' = 2$

$MC(2400) = 2$

$MR(q) = (15q^{.75})' = 15 \cdot (.75)\, q^{-1/4}$

$MR(2400) = \dfrac{45}{4(2400)^{1/4}} \approx 1.61$

d) (3 pts) Big J wants to increase production to do better this year. Based on the marginal revenue and marginal cost *at this point* ($q = 2400$), explain whether Big J's strategy is sound.

Since $MR(2400) - MC(2400)$

$= \dfrac{45}{4(2400)^{1/4}} - 2 \approx -0.39 < 0;$

~~since~~ it does <u>not</u> make sense to increase production.

13. (10 pts) In the late 1970's, complex political pressures caused the price of refined gold to rise sharply. Let t be the time in years since 1900, and let H be the worldwide gold holdings (the total amount of gold worldwide), in *tons*. At $t = 79$, the world economy held 26,000 tons of gold. This quantity was steadily growing at approximately 500 tons per year.

Also, let P be the market price of gold in *millions of dollars per ton*. For times near 1979, the price P of gold was approximately:

$$P = 9.9 + 11(t - 79)$$

a) (2 pts) How fast was the price of gold rising in 1979?

$P'(79) = 11$ (units = millions of \$/ton per year)

b) (2 pts) Let V be the dollar value of all the gold in the world. Calculate V in 1979. Include units in your answer.

$V(79) = H(79) \cdot P(79) = 26{,}000 \cdot 9.9$

$= \$\ 257{,}400$ million.

c) (6 pts) How fast was V changing in 1979? Answer in a meaningful sentence, with units.

We are told $H'(79) = 500$ tons/year.

Using the ~~chain~~ product rule:

$$V'(79) = H'(79) \cdot P(79) + H(79) \cdot P'(79)$$

$= 500 \cdot 9.9 + 26{,}000 \cdot 11$

$= 4950 + 286{,}000$

$= 290{,}950$

(units = millions of \$ per year)