Solution Guide

MATH 115 Midterm Exam I

DEPARTMENT OF MATHEMATICS
University of Michigan

October 9, 2002

NAME: Key ID NUMBER: ______

SIGNATURE: ______

INSTRUCTOR: SECTION NO:

General Instructions: Do not open this exam until you are told to begin. The exam consists of 11 questions on 9 pages (including this cover sheet). The exam is worth 100 points.

Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.

Show an appropriate amount of work for each exercise so that graders can see not only the answer but also how you obtained it. Unless explicitly stated, no credit will be given for answers that do not show how they were derived. If you use graphs or tables to obtain an answer, be certain to provide an explanation (and a sketch of the graph, if that is the method) to make it clear how you arrived at your solution. Use units where appropriate.

You are allowed two sides of a 3 by 5 card of notes and are expected to use your calculator. No other books or papers are allowed.

PROBLEM	POINTS	SCORE
1	6	
2	8	
3	4	
4	9	
5	8	
6	12	
7	11	
8	4	
9	12	
10	10	
11	16	
TOTAL	100	

1. (6 points) The table gives the values of a function f.

x	2	4	6	8
$f(x)$	15	9	6	2

(a) If f could be a linear function, find a possible formula for f. If not, explain why not.

The function is not linear because $\frac{\Delta y}{\Delta x}$ is not constant.

Note: $\frac{15-9}{-2} = -3 \neq \frac{9-6}{-2} = \frac{-3}{2}$

(b) If f could be an exponential function, find a possible formula for f. If not, explain why not.

$\frac{9}{15} = \frac{3}{5} \neq \frac{6}{9} = \frac{2}{3} \neq \frac{2}{6} = \frac{1}{3}$

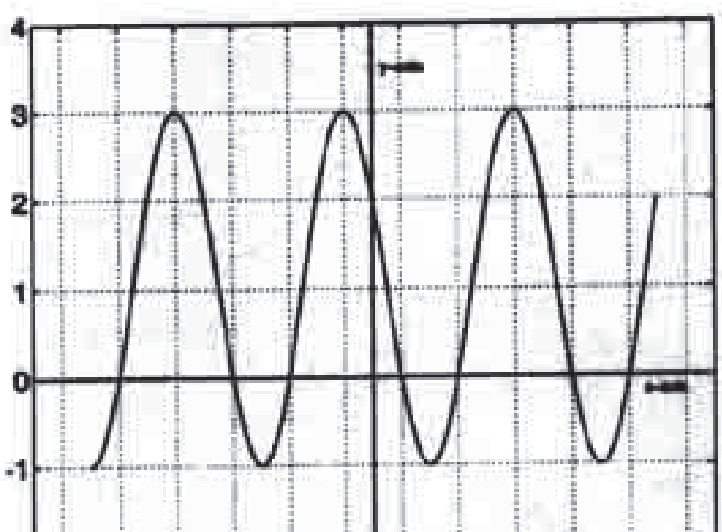
The function is not exponential because ratios of y-values over equally-spaced x-values are not constant -- ie; not a constant % rate of change.

2. (8 points) For the periodic function with the graph given below, determine:

(a) the period of the function; 6

(b) the amplitude of the function; 2

(c) a possible formula for the function. $f(x) =$ $2\cos(\pi/$ ...



3. (4 points) The functions f and g are defined for all real values of x, and g has an inverse. Although f and g are defined for all real numbers x, we have listed only a partial table of values.

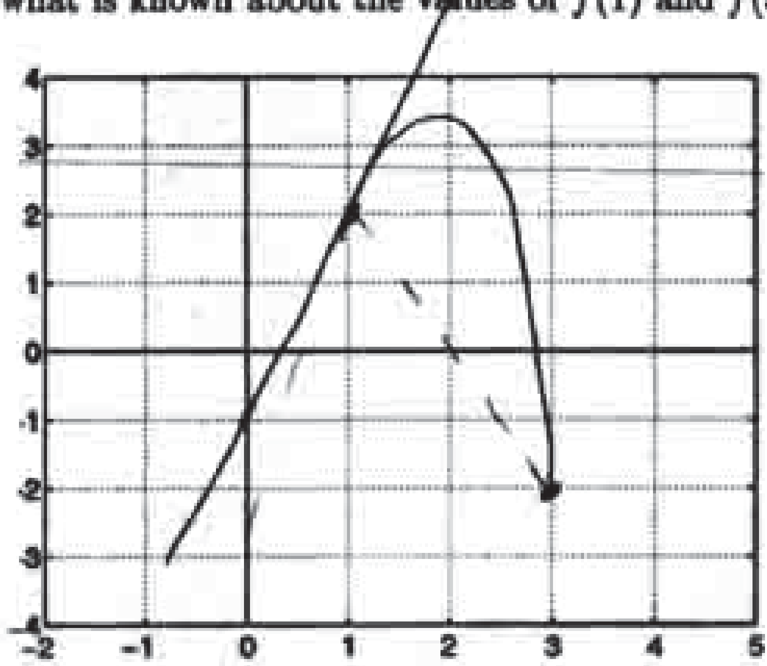
x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	1	0	-5	2	0	2	6	3	1
$g(x)$	4	2	1.5	0	-1	-1.5	-2	-3	-3.5

The following table gives some of the values of g^{-1} and the composition of g and f. Using the data given in the above table, fill in the blanks in the table below. If there is not enough information to determine the exact value, mark an "X" in the box.

x	-4	-3	-2	-1	0	1	2	3	4
$g(f(x))$	-1.5	-1	X	-2	-1	-2	X	-3	-1.5
$g^{-1}(x)$	5	3	2	0	-1	X	-3	-3.5	-4

4. (9 points) You are given that a function f has the properties that $f(1) = 2$, $f'(1) = 3$ and that the average rate of change of f on the interval from $x = 1$ to $x = 3$ is -2.

(a) Sketch a possible graph of f on the given axes. Be sure that your graph shows clearly what is known about the values of $f(1)$ and $f(3)$.



(b) Give a formula for the tangent line to the graph of f at $x = 1$. Sketch this line on your graph.

Slope = 3 point (1, 2)

$y - 2 = 3(x-1)$

$y = 3x - 1$

(see graph)

5. (8 points) (a) Find a value of k so that the function

$$f(x) = \begin{cases} 1 - x, & \text{if } x < 3; \\ kx - 4k, & \text{if } x \geq 3. \end{cases}$$

is continuous on every interval.

The function is continuous for all $x \neq 3$ since both functions are linear. We need them to meet at $x = 3$. Thus,

$1 - 3 = k(3) - 4k \rightarrow -2 = -k$, so $k = 2$

(b) Is the function you found differentiable at $x = 3$? Explain why or why not.

As $x \to 3^-$, the slope of f is -1, but as $x \to 3^+$, the slope is 2 (since for $x \geq 3$, $f(x) = 2x - 8$) There is a sharp corner @ $x = 3$. Thus, f is not differentiable at $x = 3$. There is a corner in the graph

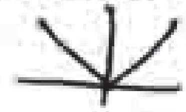
6. (12 points) Are the given statements true or false? Give an explanation for each answer.

(a) If the graph of a function g is obtained by shifting the graph of a function f vertically upward by 3 units, then $g' = f' + 3$.

False. If $g(x) = f(x) + 3$, then $g'(x) = f'(x)$. A vertical shift does not change the slope of f.

(b) If a function is not differentiable then it is not continuous.

False. For example, $y = |x|$ is continuous at $x = 0$ but not differentiable there.



(c) If $f'' > 0$, then f is increasing.

False. For example, $y = e^{-x}$ is decreasing, but $f'' > 0$.

(d) The inequality $\sqrt{x} < 2\log(x^4)$ holds for large positive values of x (that is, as $x \to +\infty$).

False. As $x \to \infty$, $\sqrt{x}$ (or any positive power of x) will overtake the log function -- no matter what the co-efficient of the log (or positive power of x ...).

7. (11 points) Over a jump site (a level field) on a particular day, parachutists know that the temperature $T = f(h)$ in degrees Celsius is given (approximately) as a function of the height h in meters above the ground. Interpret the following in practical terms, giving units.

(a) $f(1000) = 24$

At 1000 meters above the ground, the temperature is 24°C.

(b) $f^{-1}(18) = 2500$

When it is 18°C, we are 2500 meters above the ground.

(c) $f'(2000) = -.0044$

At 2000 meters, the temperature is decreasing at the rate of approximately .0044 °C per meter.

8. (4 points) Circle the answer that best describes the conditions on the first and second derivatives of the function P, where $P(t)$ is the price of gasoline at time t and the price is:

(a) rising "faster and faster"

(*i*) $P'(t) > 0$ and $P''(t) > 0$; (*ii*) $P'(t) > 0$ and $P''(t) < 0$;

(*iii*) $P'(t) < 0$ and $P''(t) > 0$; (*iv*) $P'(t) < 0$ and $P''(t) < 0$;

(b) "close to bottoming out"

(*i*) $P'(t) > 0$ and $P''(t) > 0$; (*ii*) $P'(t) > 0$ and $P''(t) < 0$;

(*iii*) $P'(t) < 0$ and $P''(t) > 0$; (*iv*) $P'(t) < 0$ and $P''(t) < 0$;

9. (12 points) **(a)** Give the formula that defines the derivative of a function f at a point a.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(b) Using the definition of the derivative, write the formula for $f'(1)$ if $f(x) = (4+x)^x$

$$f'(1) = \lim_{h \to 0} \frac{(4+1+h)^{1+h} - (4+1)^1}{h}$$

$$= \lim_{h \to 0} \frac{(5+h)^{(1+h)} - 5}{h}$$

(c) Numerically approximate $f'(1)$ correct to at least three decimal places. To receive full credit, you must show the calculations you used to justify your answer.

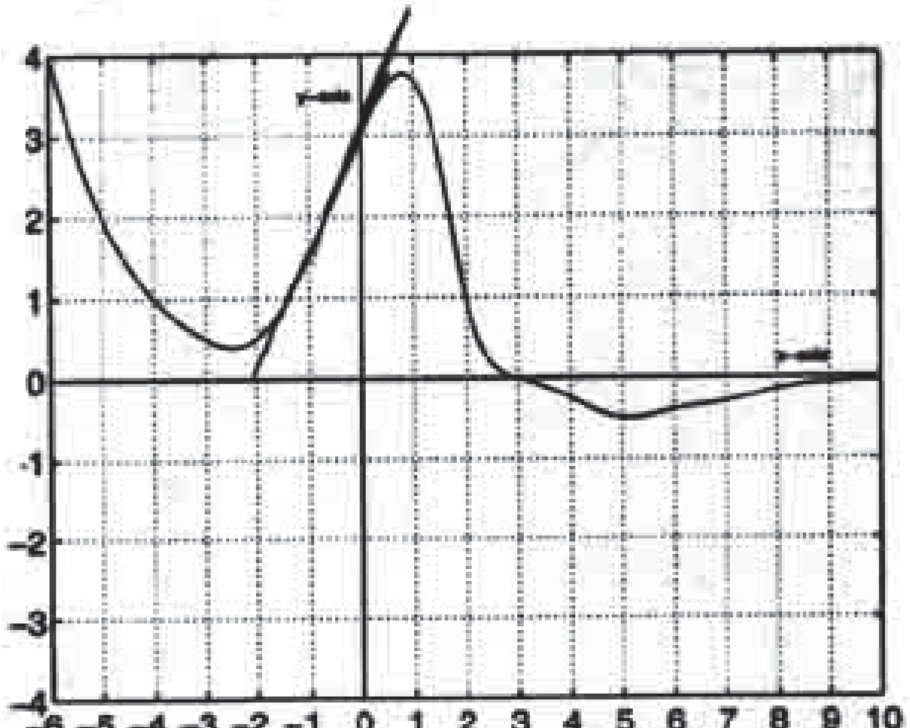
For 'small' values of 'h' we have

h	$\frac{(5+h)^{(1+h)} - 5}{h}$
.01	9.1387
.001	9.0563
.0001	9.0481
.00001	9.0473
$-.00001$	9.0471
$-.0001$	9.0463
$-.001$	9.0381
.01	8.957..

to 3 dec. places

$f'(1) \approx 9.047$

10. (10 points) The figure below gives the graph of the derivative f' of a function f



Graph of the derivative of f (not f)

(a) On what interval(s) is f increasing?

The function f is increasing for $x<3$. [or $-6<x<3$]

(b) On what interval(s) is f concave down?

The function is concave down when f' is decreasing -- i.e. for $x<\sim -2.5$ and for $1<x<5$. [or $-6<x<-2.5$ and $1<x<5$]

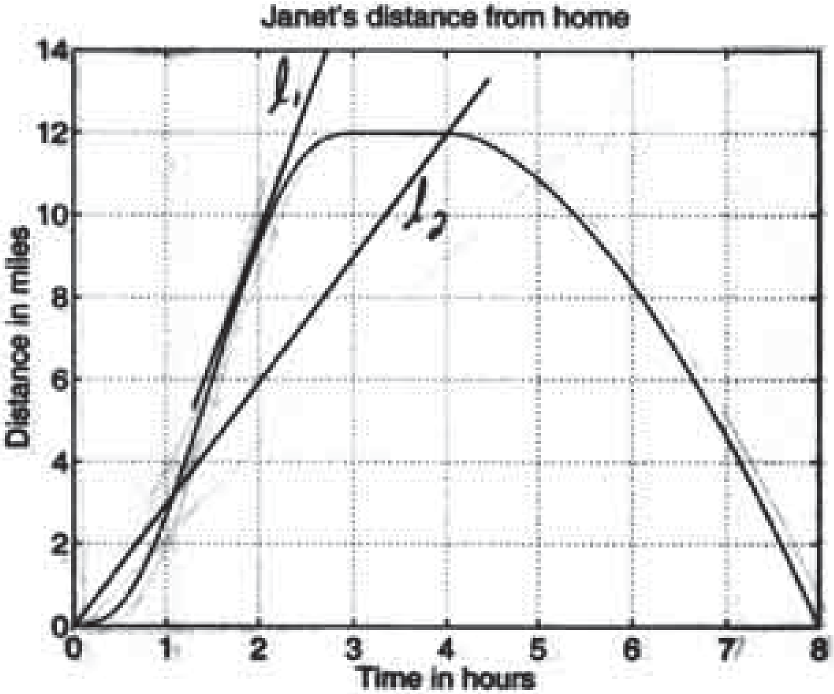
(c) For what value of x (approximately) is $f(x)$ the largest? Explain.

The function is increasing until $x=3$ & then decreases. Thus, f has its largest value @ $x=3$.

(d) For what value of x (approximately) is $f''(x)$ the largest? Explain.

The second derivative, f'' would be largest when the slope of f' (or a tangent to f') is steepest in a positive direction. This appears to be around $x \approx -1$. (see graph.)

11. (16 points) Janet rides her bicycle on a day trip (8 hours) along a straight north-south road. Her distance $s(t)$ in miles north of her home t hours after her trip begins is given by the following graph.



(a) Which is larger? Janet's average velocity for the first four hours or her instantaneous velocity two hours after the start of the trip? Explain.

Janet's instantaneous velocity two hours after the start is larger than the average for the first 4 hours. The slope of the line l_1 is greater than l_2. (See graph.)

(b) Did Janet stop during her trip? Explain.

Yes. The interval from about $t=3$ to $t=4$ indicates that her distance is neither increasing or decreasing. Her velocity is zero. Janet is stopped.

(c) Approximately when after the start of the trip is Janet riding the fastest? Explain.

This would occur when her velocity is greatest & therefore the slope of the graph (or tangent to the graph) is greatest. This appears to be at around $t \approx 1.5$ hrs. (see above)

(d) Are there any time intervals over which Janet's acceleration is positive? If so, which? Explain why you know this.

Yes. Acceleration is positive for about the first 1.5 hours. This can be seen on the graph above where the function is concave up. Over that interval, f' (or velocity) is increasing

Continuation of problem 11

(e) On the set of axes provided here, draw a graph of Janet's velocity. Be sure to label relevant axes with appropriate units and select an appropriate numerical scale for them. To help you in sketching the graph, another copy of the graph of $s(t)$ is included below the axes where you should sketch your graph of Janet's velocity.

11. (e)