MATH 115 MIDTERM EXAM # 2

DEPARTMENT OF MATHEMATICS
University of Michigan

November 13, 2002

NAME: ______________________ ID NUMBER: Key

SIGNATURE: ________________________________

INSTRUCTOR: ________________ SECTION NO: ______________________

General Instructions: Do not open this exam until you are told to begin. This test consists of 9 problems on 8 pages (including this cover sheet). The exam is worth 100 points. Do not separate the pages of exam. If any pages do become detached, write your name on them and point them out to your instructor when you turn in the exam.

Please read the instructions for each individual exercise carefully. Show an appropriate amount of work for each exercise so that graders can see not only the answer but also how you obtained it. If you use graphs or tables to obtain an answer, be certain to provide an explanation (and a sketch of the graph, if that is the method) to make it clear how you arrived at your solution. Use units where appropriate.

You are allowed two sides of a 3 by 5 card of notes and are expected to use your calculator.

PROBLEM	POINTS	SCORE
1	12	
2	8	
3	9	
4	9	
5	10	
6	11	
7	12	
8	15	
9	14	
TOTAL	100	

This page contains short answer questions. No explanations are required.

1. (12 points) .

(a) Compute the 25th derivative, $f^{(25)}$, of the function f given by

(i) $f(x) = 10x^9 + 14x^7 - 12x^6 + 2x^5 + 3x^4 - 2x^2 + 5x - 4$

$f^{(25)}(x) =$ ___0___

(ii) $f(x) = \sin(2x)$

$f^{(25)}(x) =$ ___$2^{25}\cos(2x)$___

(b) For what value of a is $\lim_{h\to 0}(a^h - 1)/h$ equal to 1?

$a =$ ___e___

(c) For the function $f(x) = (1.2)^{3x}$, find

(i) $f'(2/3) =$ ___$3\ln(1.2)(1.2)^2$___ $(\approx .787639..)$

(ii) $[f(2/3)]' =$ ___0___

2. (8 pts) The function f is an increasing function that is concave down. Fill in each of the blanks with one of the symbols, $<, =, >$ so that the following statments about f are always true.

(i) $f(2)$ ___$<$___ $f(4)$

(ii) $f'(2)$ ___$>$___ $f'(4)$

(iii) $f''(2)$ $<$ 0

(iv) $f(3 + \Delta x)$ ___$<$___ $f(3) + f'(3)\Delta x$

3. (9 points)[Show your work.] Use the information given in the table to find $h'(4)$ if:

x	1	2	3	4
$f(x)$	2	1	4	2
$f'(x)$	3	2	-1	2
$g(x)$	4	2	1	3
$g'(x)$	3	2	2	-3

$$h'(x) = \frac{g'(x)f(x) - g(x)f'(x)}{(f(x))^2}$$

(i) $h(x) = g(x)/f(x)$; $h'(4) =$ -3

$$h'(4) = \frac{(-3)(2) - 3(2)}{4} = \frac{-12}{4} =$$

(ii) $h(x) = f(\sqrt{x})$; $h'(x) = f'(\sqrt{x}) \cdot \frac{1}{2} x^{-1/2}$ $=$ $h'(4) =$ $1/2$

$$h'(4) = f'(2) \cdot \frac{1}{4} = 2\left(\frac{1}{4}\right)$$

(iii) $h(x) = \ln(g(x))$; $h'(4) =$ -1

$$h'(x) = \frac{1}{g(x)} \cdot g'(x)$$

$$h'(4) = \frac{1}{3}(-3) = -1$$

4. (9 points) On what interval(s) is the function $f(x) = e^{-x^4}$ increasing and concave down? [Show your work.]

$$f'(x) = -4x^3 e^{-x^4}$$

$f'(x) > 0$ when $x < 0$

$$f''(x) = -12x^2 e^{-x^4} + (-4x^3)e^{-x^4}(-4x^3)$$

$$= e^{-x^4}(4x^2)(4x^4 - 3)$$

$f''(x) < 0$ when $4x^4 < 3$
$x^4 < \frac{3}{4} \rightarrow |x| < \sqrt[4]{\frac{3}{4}}$
or $-\sqrt[4]{\frac{3}{4}} < x < \sqrt[4]{\frac{3}{4}}$

Thus,

ANSWER: f is increasing and concave down on the interval(s): $\left(-\sqrt[4]{\frac{3}{4}}, 0\right)$

5. (10 points) The *trebuchet*, a medieval catapult driven by a falling, hinged counterweight, can be simulated with the use of mathematical models. The range of the projectile flung from the catapult at an angle θ is given by

$$R = \frac{2v_0{}^2 \sin\theta\cos\theta}{g},$$

where g is the constant acceleration due to gravity and v_0 is the constant representing the initial velocity of the projectile.

(a) Find the exact value of θ on the interval $0 \le \theta \le \pi/2$ that maximizes the range of the projectile.

$$R' = \frac{2v_0^2}{g}\left(\cos\theta\cos\theta + \sin\theta(-\sin\theta)\right)$$

$$= \frac{2v_0^2}{g}\left(\cos^2\theta - \sin^2\theta\right)$$

$R' = 0$ if $\cos^2\theta = \sin^2\theta$ or $\tan^2\theta = 1$

Thus, on $\left[0, \frac{\pi}{2}\right]$, $\theta = \frac{\pi}{4}$ is the only C.P.

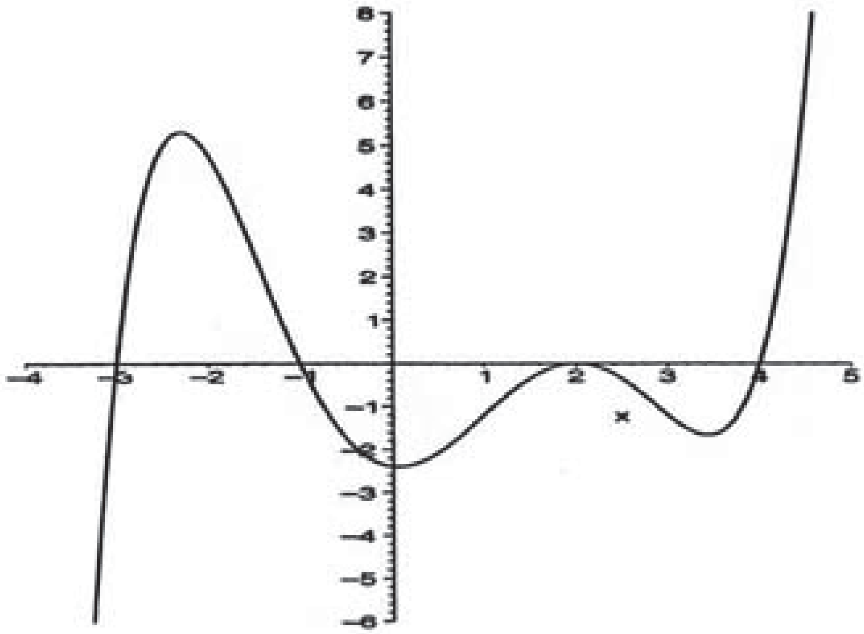
NOTE: $R(0) = 0 = R\left(\frac{\pi}{2}\right)$. Therefore, $R\left(\frac{\pi}{4}\right)$ is the max.

& $R\left(\frac{\pi}{4}\right) = \frac{V_0^2}{g} > 0$

(b) What is the maximum range?

$$R\left(\frac{\pi}{4}\right) = \frac{2v_0^2}{g}\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{V_0^2}{g}$$

6. (11 points) A function f is differentiable at all points and a graph of f', the derivative of f, is shown in the figure.



(a) List all values of x (approximately) that are critical points of f or write "none" if there aren't any.

$x =$ -3, -1, 2, 4

(b) List all values of x (approximately) where f has a relative maximum or write "none" if there aren't any.

$x =$ -1

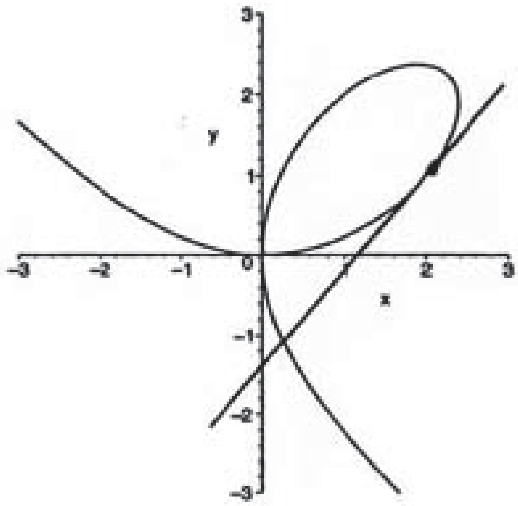
(c) List all values of x (approximately) where f has a relative minimum or or write "none" if there aren't any.

$x =$ -3, 4

(d) List all values of x (approximately) where f has an inflection point or write "none" if there aren't any.

$x =$ -2, 0, 2, 3.5

7. (12 points) The figure shows the graph of the curve $2x^3 + 2y^3 - 9xy = 0$, called a *folium of Descartes* because it was studied by Descartes in about 1638. 1



(a) Using the equation above, show that the point $(2, 1)$ lies on the curve.

$$16 + 2 - 18 \stackrel{\checkmark}{=} 0$$

(b) Compute the derivative $\frac{dy}{dx}$ of the function of x defined implicitly by the equation.

$$6x^2 + 6y^2\frac{dy}{dx} - 9y - 9x\frac{dy}{dx} = 0$$

$$(6y^2 - 9x)\frac{dy}{dx} = 9y - 6x^2 \longrightarrow \frac{dy}{dx} = \frac{9y - 6x^2}{6y^2 - 9x}$$

(c) What is the slope of the tangent line to the curve at the point $(2, 1)$?

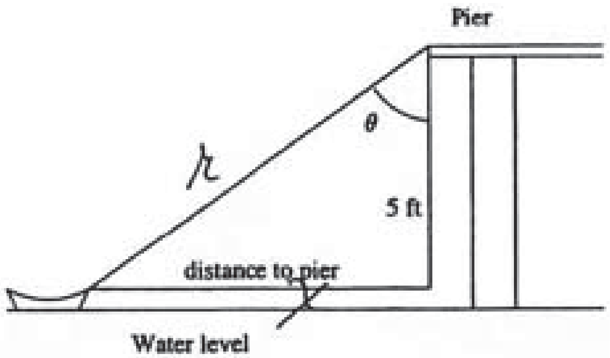
$$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{9-24}{6-18} = \frac{-15}{-12} = \frac{5}{4}$$

(d) Write the equation of the tangent line to the curve at the point $(2, 1)$. Draw the graph of this line on the figure.

$$y - 1 = \frac{5}{4}(x - 2)$$

$$y = \frac{5}{4}x - \frac{3}{2}$$

8. (15 points) A boat is pulled toward a dock by a rope from the bow through a ring on the dock 5 feet above its bow. (See figure) The rope is hauled in at a rate of 2 feet per second.



given $\frac{dr}{dt} = -2$ ft/sec

In answering the following questions, use complete sentences, show your work and use units.

(a) How fast is the boat approaching the dock when 13 feet of rope are out?

$$r^2 = x^2 + 25$$

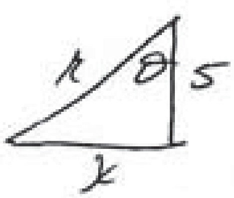
$$2r\frac{dr}{dt} = 2x\frac{dx}{dt} \rightarrow \frac{dx}{dt} = \frac{r}{x}\frac{dr}{dt}$$

when $r = 13$, $169 - 25 = x^2$ so $x = 12$ (discard -12)

Thus, $\frac{dx}{dt} = \frac{13}{12}(-2) = \frac{-13}{6}$ ft/sec.

When 13 feet of rope are out, the boat is approaching the dock at 13/6 ft per second (or $2\frac{1}{6}$ ft/sec).

(b) At what rate is the angle θ changing at that time?



$$\tan\theta = \frac{x}{5}$$

$$\frac{1}{\cos^2\theta}\frac{d\theta}{dt} = \frac{1}{5}\frac{dx}{dt}$$

at $s = 13$
$\cos\theta = \frac{5}{13}$

$$\frac{d\theta}{dt} = \frac{1}{5}\cos^2\theta\left(\frac{dx}{dt}\right) = \frac{1}{5}\left(\frac{5}{13}\right)^2\left(\frac{-13}{6}\right) = \frac{-5}{78}$$

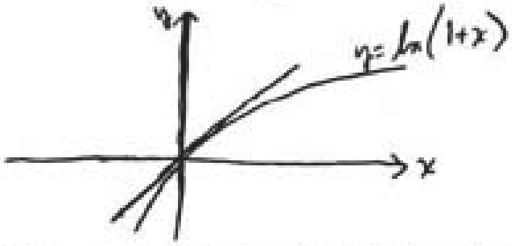
The angle is decreasing at the rate of $5/78 \approx .064$ radians per second.

9. (14 points) **(a)** Find the local linearization of the function $f(x) = \ln(1+x)$ near the point $x = 0$. Show your work.

$$f'(x) = \frac{1}{1+x} \qquad f(0) = 0 \quad f'(0) = 1$$

$$f(x) \approx 0 + x = x$$
(near $x=0$)

(b) Is the approximation to $\ln(1+x)$ given by the local linearization an underestimate or overestimate? Explain why?



The approximation is an overestimate since f is concave down

(c) We saw in Chapter 1 of the text that P_0 dollars invested at a rate of $r\%$ per year grows to be worth $P_0(1 + r/100)^t$ dollars after t years. Compute, in terms of the interest rate r, how long it takes for the invested money to double in value?

$$2P_0 = P_0\left(1 + \frac{r}{100}\right)^t \qquad t = \frac{\ln 2}{\ln\left(1 + \frac{r}{100}\right)}$$

$$\ln 2 = t \ln\left(1 + \frac{r}{100}\right)$$

(d) A common rule of thumb used by investors is the "Rule of 70" — *money invested at a $r\%$ interest per year doubles in value in $70/r$ years.* Explain why this is a reasonable approximation to the actual doubling time.

Since $\ln\left(1 + \frac{r}{100}\right) \approx \frac{r}{100}$ (from part (a)), the money doubles in approximately $\frac{\ln 2}{\frac{r}{100}} \approx \frac{.69}{\frac{r}{100}} = \frac{69}{r}$ years, and $\frac{69}{r}$ is close to $\frac{70}{r}$