MATH 115 FINAL EXAM

DEPARTMENT OF MATHEMATICS
University of Michigan

December 13, 2002

NAME: Key ID NUMBER: ____________

SIGNATURE: ____________

INSTRUCTOR: ____________ SECTION NO: ____________

General Instructions: Do not open this exam until you are told to begin. This test consists of 10 problems on 10 pages (including this cover sheet). The exam is worth 100 points. Do not separate the pages of exam. If any pages do become detached, write your name on them and point them out to your instructor when you turn in the exam.

Please read the instructions for each individual exercise carefully. Show an appropriate amount of work for each exercise so that graders can see not only the answer but also how you obtained it. If you use graphs or tables to obtain an answer, be certain to provide an explanation (and a sketch of the graph, if that is the method) to make it clear how you arrived at your solution. Use units where appropriate.

You are allowed two sides of a 3 by 5 card of notes and are expected to use your calculator.

PROBLEM	POINTS	SCORE
1	11	
2	8	
3	4	
4	15	
5	10	
6	7	
7	9	
8	12	
9	14	
10	10	
TOTAL	100	

1. 1 points) **(a)** What is the average of the function x^3 on the interval $1 < x < 3$?

$$\frac{1}{3-1}\int_1^3 x^3\,dx = \frac{1}{2}\left(\frac{x^4}{4}\right)\Big|_1^3 = \frac{1}{2}\left(\frac{3^4}{4} - \frac{1}{4}\right) = \left(\frac{80}{4}\right)\frac{1}{2} = 10$$

(b) If it is known that $\int_1^3 f(x)\,dx = 4$ and $\int_1^3 (f(x))^2\,dx = 5$, then

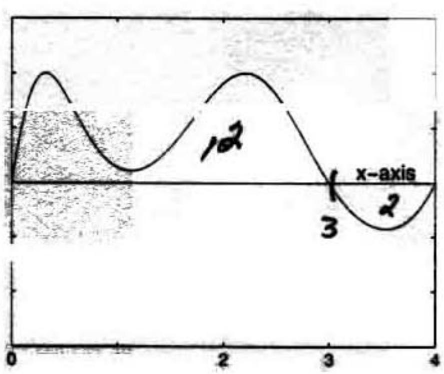
$$\int_1^3 (1 + f(x))^2\,dx = \underline{\quad 15 \quad}$$

$$\int_1^3 1\,dx + 2\int_1^3 (f(x))\,dx + \int_1^3 (f(x))^2\,dx$$

$$3-1 + 2(4) + 5 = 15$$

(c) A function $f(x)$ has a graph as shown below, and it is known that $\int_0^4 f(x)\,dx = 10$, while the area of the region below the x-axis and above the graph of f is 2. Find

$$\int_0^3 f(x)\,dx = \underline{\quad 12 \quad}$$



(d) The average price (in dollars) of a new house that is A square feet in area is a function $P = f(A)$. What are the units of $dP/dA = f'(A)$.

$$\frac{dP}{dA} = \frac{\$}{ft^2}$$

dollars per sq. foot

2. (8 points) **(a)** Give the statement of the fundamental theorem of calculus.

If f is continuous on $[a,b]$ and $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

(b) Give a specific instance of the fundamental theorem by using the interval $-2 \le x \le 3$ and the function x^2 for one of the functions in your statement of the theorem.

Example

$$\int_{-2}^{3} x^2\,dx = \frac{x^3}{3}\Big]_{-2}^{3} = \frac{3^3}{3} \quad \frac{(-2)^3}{3} = 9 + \frac{8}{3} = \frac{35}{3}$$

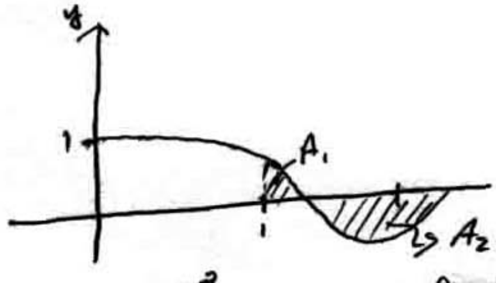
or

$$\int_{2}^{3} 2x\,dx = x^2\Big]_{-2}^{3} = 9 \quad 4 = 5$$

3. (4 points) Suppose $f'(x) = \cos(x^2)$. Use the graph of $f'(x)$ to decide which is larger, $f(1)$ or $f(2)$. Explain the reason for your answer.

graph of $\cos(x^2)$ on $[0,2]$ looks

like



Thus $f(1) > f(2)$,

since $\int_1^2 f'(x)\,dx = f(2) - f(1)$ would be negative

i.e. the small positive A_1 minus the larger A_2

4. (15 points) **(a)** Given that $g(x)$ $f(e^{-x})$, where f is a function with $f'(1) = 3$ and $f''(1) = -5$, compute $g'(0)$ and $g''(0)$.

ANSWERS: $g'(0) =$ <u>-3</u> $g''(0) =$ <u>-2</u>

$$g(x) = f(e^{-x});\quad g'(x) = f'(e^{-x})\cdot e^{-x}(-1)$$

$$g'(0) = f'(1)(1)(-1) = 3(-1) = -3$$

$$g''(x) = f''(e^{-x})(-e^{-x})(e^{-x})(-1) + f'(e^{-x})e^{-x}$$

$$g''(0) = f''(1)(-1)(1)(-1) + (3)(1) = -5+3 = -2$$

(b) Show that the point $x =$ $y = \pi/4$ lies on the curve

$$2 + xy = \frac{\pi}{4} + x^2 + \tan(y) \longrightarrow 2 + \frac{\pi}{4} = \frac{\pi}{4} + \quad + \quad = \frac{\pi}{4} + 2$$

and calculate dy/dx at this point

$$x\frac{dy}{dx} = 2x + \frac{1}{\cos^2 y}\frac{dy}{dx}$$

when $x = 1$, $y = \frac{\pi}{4}$

$$\frac{\pi}{4} + \frac{dy}{dx} = 2 + \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2}\frac{dy}{dx} \longrightarrow \frac{dy}{dx}(2-1) = \frac{\pi}{4} - 2$$

$$\frac{dy}{dx} = \frac{\pi}{4} - 2$$

(c) The cost function $C(q)$ represents the cost in dollars of productng q units of some good and the revenue function $R(q)$ represents the revenue in dollars received by selling q units of the good. If $C'(500) = 100$ and $R'(500) = 125$, should the quantity produced be increased or decreased from $q = 500$ in order to increase profits? Explain the reason for your answer.

yes

$$\pi(q) = R(q) - C(q)$$

$$\pi'(q) = R'(q) - C'(q)$$

$$\pi'(500) = 125 - 100 = 25$$

Profit is increasing @ $q = 500$

$\pi'(500) = 25$ essentially means an additional unit will bring in an extra \$25 in profit

5. 10 points)

The above graph describes the revenue of Mammoth Corporation as a function of its advertising budget, expressed in hundreds of millions of dollars.

(a) Interpret the meaning of the y-intercept of the graph in the context of this problem.

$R(0) = 2$ indicates that without any advertising the company's revenue is 2 hundred million dollars.

(b) What does the concavity of the graph indicate in terms of revenue and the amount spent on advertising?

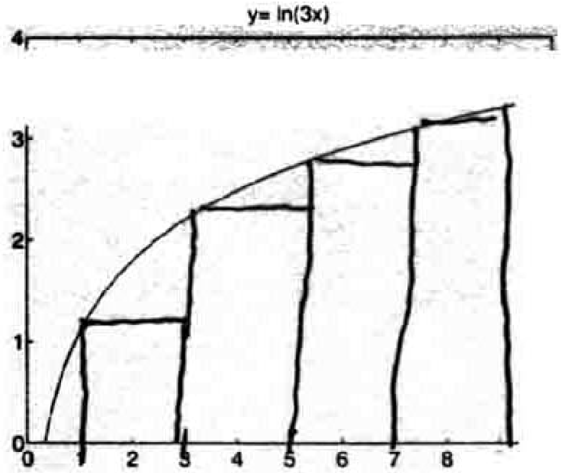
Although revenue is increasing, it is increasing at a slower & slower rate as advertising increases.

(c) What does the statement $R'(4) = .297$ mean in practical terms?

After \$400 million on ads are spent, revenue will increase by only approx \$29.7 million for the next \$100 million in ads.

(d) Assuming the graph gives a valid estimate of the company's revenues as a function of money spent on advertising, should they spend as much as possible on advertising? If so, explain why. If not, explain why not and give your best approximation for the amount the company should spend.

No. Once $R' < 1$, spending more on ads means for each dollar spent on ads, the company takes in less than a dollar on increased revenue. Looks like the company should spend ~ \$100 million on advertising (slope @ $x=1$ is approx. 1).



6. (7 points) The graph shown above is of the function $f(x) = \ln(3x)$.

(a) Without computing a final answer (i.e. do not use your calculator), give an exact expression for the left hand sum using four subdivisions that estimates the integral of f on the interval $1 \le x \le 9$.

$$LHS_{(4)} = (\ln 3 + \ln 9 + \ln 15 + \ln 21)(2)$$

$$\Delta x = \frac{9-1}{4} = 2$$

(b) Illustrate the sum on the graph

(c) Would your sum be an underestimate or an overestimate of $\int_1^9 \ln(3x)\,dx$? Explain why.

The LHS is an underestimate because the function is increasing on [1, 9].

$f(x) > 0 \quad f'(x) < 0 \quad f''(x) < 0$

7. (9 points) Let f be a function that is positive, decreasing, and concave down for $1 < x < 2$. Let g be the function defined by $g(x) = 1/f(x)$. Use the methods of calculus to show why the statements in **(a)** and **(b)** are true.

(a) g is increasing for $1 < x < 2$.

$$g(x) = \frac{1}{f(x)} \longrightarrow g'(x) = \frac{-f'(x)}{(f(x))^2} \longrightarrow \frac{--}{+} = \frac{+}{+}$$

(b) g is concave up for $1 < x < 2$.

$$g''(x) = \frac{(f(x))^2(-f''(x)) + f'(x)2(f(x))f'(x)}{(f(x))^4}$$

$$\longrightarrow \frac{(+)(-)(-) + (-)(+)(-)}{+} = \frac{+ + +}{+} = +$$

$f > 0 \quad f' < 0 \quad f'' > 0$

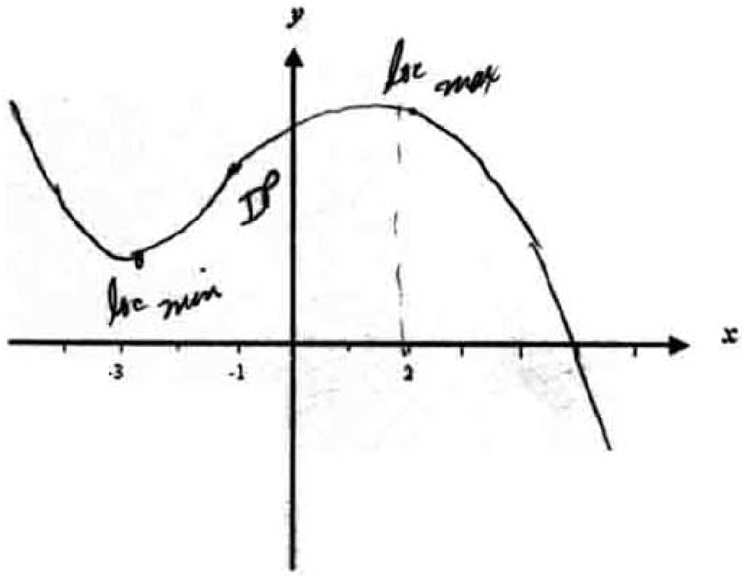
(c) If f is positive, decreasing, and *concave up* for $1 < x < 2$, is $g(x) = 1/f(x)$ always increasing and concave down for $1 < x < 2$? Explain why or why not.

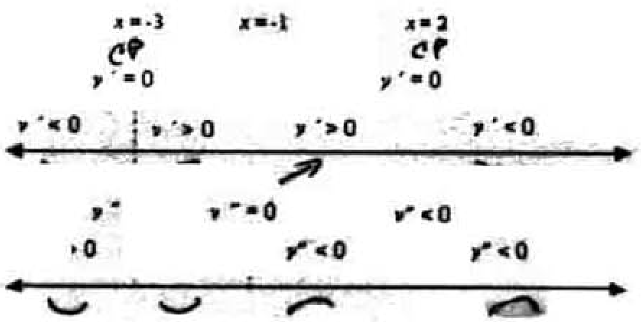
$$g'(x) = \frac{f'(x)}{(f(x))^2} \longrightarrow \frac{--}{+} = \frac{+}{+} \rightarrow \text{still increasing}$$

$$g''(x) \Rightarrow \frac{+(-)(+) + (-)(+)(-)}{+} \rightarrow \frac{(-)+(+)}{+}$$

Cannot say for sure where g is conc. up or concave down.

8. (12 points) **(a)** On the axes provided, skech a possible graph of $y = f(x)$ using the given information about the derivatives $y' = f'(x)$ and $y'' = f''(x)$.



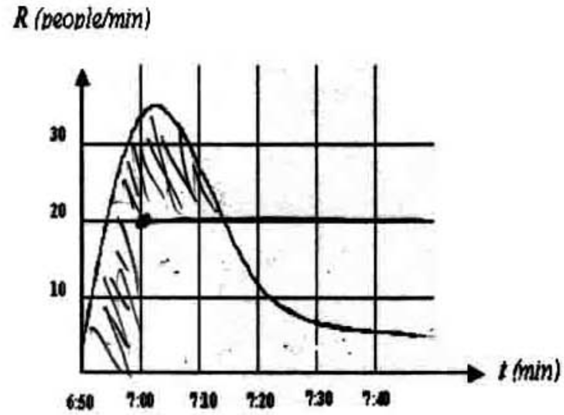


(b) List all values of x where f has, or could have, a local minimum, local maximum, or an inflection point.

Local Minimum $x = -3$ Local Maximum $x = 2$ Inflection Point(s) $x = -1$

(c) (Still refering to the function of part (a)). How many distinct zeros COULD there be of a function with the properties in part (a)? (Circle ALL correct answers).

None 1 2 3 4 5 6 8 9 infinitely many



9. (14 points) On the day after Thanksgiving, the stores were mobbed with shoppers. In the local ToysWereU Store there were already 50 people in line when the security guards showed up at 6:50 a.m. The graph above shows the rate, R, in arrivals/minute at which people arrived after 6:50.

The store opens at 7:00 a.m., and the guards are to allow people into the store at a constant rate of 20 people per minute. Use this information and the graph to estimate the following:

(a) The length of the line (i.e. the number of people) at 7:00 when the guards begin letting people into the store.

$50 + 200 \approx 250$ people

(b) The length of the line at 7:20.

$$\underbrace{250}_{\text{at } 7:00} + \int_{7:00}^{7:20} R(t)\,dt - \underbrace{400}_{\text{ppl } \text{l'd}} \approx \underbrace{250 + 500}_{750} - 400 \approx 350 \text{ ppl.}$$

(c) The rate at which the line is growing at 7:10.

$$\sim 27 \frac{\text{ppl}}{\text{min}} - 20 \frac{\text{ppl}}{\text{min}} \sim 7 \frac{\text{ppl}}{\text{min}}$$

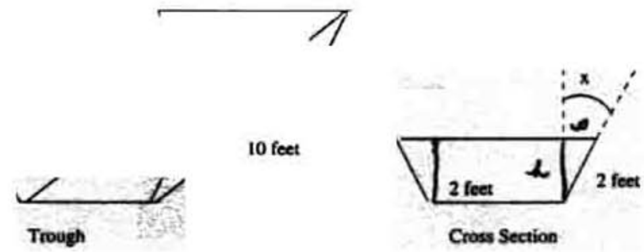
(d) The length of time a person who arrives at 7:00 has to stand in line.

$$\frac{250}{20} = \sim 12.5 \text{ minutes}$$

(e) The time at which the line disappears.

By 7:40, ~900 ppl have arrived (including the original 50). 800 have been let into the store. The line is decreasing @ ~15 ppl/min → So ~ 7:47 the line should disappear.

10. (10 points) A trough, as shown in the figure, is to be made with a base that is 2 feet wide and 10 feet long. The sides of the trough are also 2 feet wide by 10 feet long and are to be placed so that they make an angle x with the vertical.



$\frac{h}{2} = \cos x$ $\quad h = 2\cos x$

$\frac{w}{2} = \sin x$ $\quad w = 2\sin x$

(a) What is the area, in terms of x, of a cross section of the trough perpendicular its long side? What is the volume of the trough? Show your work.

$$A(x) = 2(2\cos x) + 2\left(\frac{1}{2}(2\sin x)(2\cos x)\right) = 4\cos x + 4\cos x \sin x \text{ ft}^2$$

$$V(x) = 10(4\cos x + 4\cos x \sin x) \text{ ft}^3$$

(b) What angle x will give the trough of largest volume, and what is that volume? Explain how you found your answer, along with any supporting evidence (you may use your calculator).

$$V'(x) = 40\left(-\sin x + \cos x(\cos x) - \sin x \sin x\right)$$

$$= 40 \quad -\sin x + \cos^2 x - \sin^2 x)$$

$$= 40\left(-\sin x + 1 - \sin^2 x - \sin^2 x\right)$$

$$40\left(1 - \sin x - 2\sin^2 x\right)$$

$$V' = 0 \text{ if } (1 - 2\sin x)(1 + \sin x) = 0$$

$$\rightarrow \sin x = \frac{1}{2} \text{ or } \cancel{\sin x = -1}$$

can be excluded

It makes sense to restrict x, $0 \le x \le \frac{\pi}{2}$, so CP on that interval is @ $x = \frac{\pi}{6}$.

$V(0) = 40$

$V(\pi/2) = 0$

$+ \; V\left(\frac{\pi}{6}\right) \approx 51.96$

So max occurs when $x = \pi/6$ & max volume is $\approx 52 \text{ ft}^3$.