MATH 115 — FIRST MIDTERM EXAM

October 8, 2003

NAME: SOLUTION KEY

INSTRUCTOR: ____________________ SECTION NO: ______________

1. Do not open this exam until you are told to begin.
2. This exam has 9 pages including this cover. There are 10 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed 2 sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones.

PROBLEM	POINTS	SCORE
1	16	
2	5	
3	6	
4	11	
5	12	
6	6	
7	8	
8	10	
9	14	
10	12	
TOTAL	100	

The questions on this page are multiple choice. They do not require an explanation. For each question, circle your choice for the correct answer(s). (4 points each part. No partial credit.)

(1A.) Let $V(t)$ represent the number of thousands of gallons of water in a tank t hours after midnight on a fixed day. Circle the pair of equations below that expresses the following statement:

"At 3PM there were 9000 gallons of water in the tank, and the amount of water in the tank was decreasing at the rate of 200 gallons per hour."

(a) $V(3) = 9000$, $V'(3) = -200$

(b) $V(3) = 9$, $V'(3) = 200$

(c) $V(15) = 9000$, $V'(15) = -200$

$\boxed{\text{(d) } V(15) = 9,\ V'(15) = -0.2}$

(e) $V(15) = 9$, $V'(15) = 0.2$

(1B.) As $x \to \infty$, the function $f(x) = \frac{x^3+3x+5}{2x^3-7x+6}$ approaches

(a) $y = 0$

$\boxed{\text{(b) } y = \frac{1}{2}}$

(c) $y = \frac{5}{6}$

(d) $y = 1$

(e) ∞

(1C.) If the product $f(x_0) \cdot f'(x_0) \cdot f''(x_0) > 0$, then which of the following is possible? Circle any answer(s) which could be true:

(a) The graph is above the x-axis, and is decreasing and concave up.

$\boxed{\text{(b) The graph is above the } x\text{-axis, and is decreasing and concave down.}}$

(c) The graph is above the x-axis, and is increasing and concave down.

(d) The graph is below the x-axis, and is increasing and concave up.

(e) The graph is below the x-axis, and is decreasing and concave down.

(1D) Suppose that $\lim_{h \to 0} \frac{f(2+h)-f(2)}{h} = 1$. Which of the following must be true?

1. $\lim_{x \to 2} f(x) = 1$.
2. f is continuous at $x = 2$.
3. $f'(2) = 1$.

(a) 1 only

(b) 2 only

(c) 3 only

$\boxed{\text{(d) 2 and 3 only}}$

(e) 1, 2, and 3

(2.) (5 points) Suppose f is a function that satisfies the following three properties:

1. f is a power function.
2. $f(1) = -7$.
3. $f(2x) = 8f(x)$, for every x.

Determine the exact formula for $f(x)$.

$$f(x) = kx^p$$
$$k = f(1) = -7$$
$$f(2x) = (-7)2^p x^p, 8f(x) = 8(-7)x^p$$
$$2^p = 8, p = 3$$
$$\boxed{f(x) = -7x^3}$$

(3.) (6 points) Let $f(x) = \frac{x^2-1}{x-1}$, and let $g(x) = x + 1$.

(a) Are f and g the same function? Why or why not?

No. Although f and g agree whenever $x \neq 1$, the two functions do not agree when $x = 1$.

(b) Let h be the function whose output is always 4, except that $h(-1)$ is undefined. Write a formula for $h(x)$ as a rational function.

$$h(x) = \frac{4(x+1)}{x+1}$$

(4.) (11 points) The table below gives the approximate number of cell phone subscribers, S, in millions, worldwide.

Year	1995	1996	1997	1998	1999	2000
Subscribers	91	138	210	320	485	738

(a) Demonstrate using the data in the table why it makes sense to model this data with an exponential function. Explain your reasoning.

Exponential functions are characterized by a constant growth factor. Note that the time intervals in the table are all the same (1 year), so we can check that the growth rate is constant by computing the ratios of consecutive output values. We see that

$$\frac{138}{91} \approx \frac{210}{138} \approx \frac{320}{210} \approx \frac{485}{320} \approx \frac{738}{485} \approx 1.52.$$

Thus, since the ratio is always approximately equal to 1.52 for equally-spaced time intervals, an exponential model is reasonable.

(b) Write a formula for S as an exponential function of t, the number of years since 1995.

$$S(t) = 91(1.52)^t$$

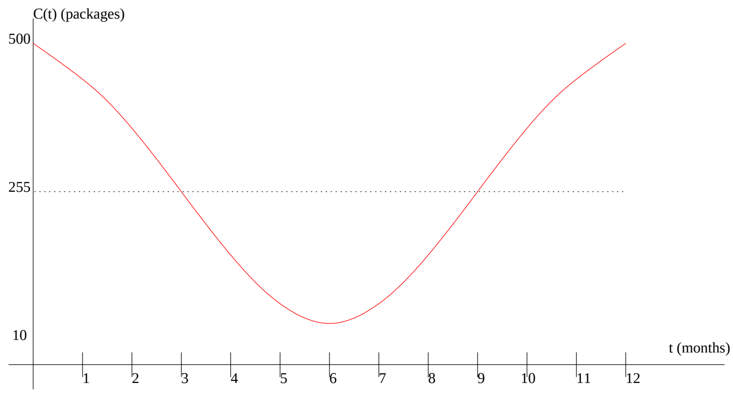
(c) In a single sentence describe how the number of cell phone subscribers has been changing since 1995. Use everyday words and do not use the symbols S or t.

Since 1995, the number of cell phone subscribers has been increasing by approximately 52% per year.

(5.) (12 points) Americans have peculiar food allegiances. For trick-or-treat season, the empty calorie of choice is candy corn – tiny "kernels" made of sugar and food coloring. The sweet, which made its debut in the 1920s, is the top-selling non-chocolate Halloween candy in the US. Unfortunately for connoisseurs of candy corn, the supply is seasonal – it's much easier to find candy corn in mid-October than it is in mid-April, for example.

Let $C(t)$ be the number of bags of candy corn on the shelves of your local grocery store, where t is the number of months since mid-October. Assume that C is periodic with a period of one year, reaching a maximum of 500 bags in mid-October, and a minimum of 10 bags in mid-April (despite its virtues, candy corn is not a popular spring-season candy).

(a) On the axes below, draw a graph of C as a function of t, in months, where $t = 0$ represents October 15. Be sure to label your axes.



(b) Determine a formula for $C(t)$.

$$C(t) = 245\cos\left(\frac{2\pi}{12}t\right) + 255$$

(c) Approximately what month during the year is the candy corn supply increasing the fastest?

The slope of the cosine curve is greatest three-quarters of the way through its period – in this case, when $t = 9$. This corresponds to nine months after mid-October, which is mid-July.

(6.) (6 points) Let $f(x) = x^{3x}$. Use the **definition** of the derivative to express $f'(2)$ as a limit. You do not need to simplify your expression or try to estimate $f'(2)$.

$$f'(2) = \lim_{h \to 0} \frac{(2+h)^{3(2+h)} - 2^6}{h}$$

(7.) (8 points) Suppose g is a differentiable function that satisfies the following three properties:

1. g is concave up.
2. $g(1) = 9$.
3. $g(5) = 3$.

(a) What is the average rate of change of g on the interval $[1, 5]$?

$$\frac{3-9}{5-1} = -\frac{6}{4} = \boxed{-\frac{3}{2}}$$

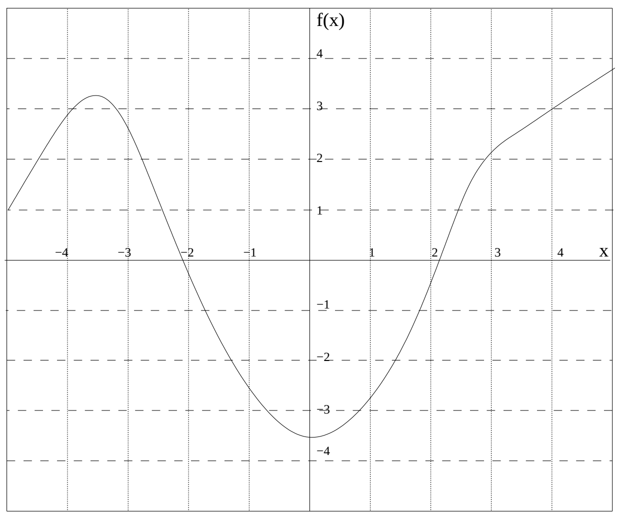
(b) Which is larger, $g'(2)$ or $g'(4)$? Explain.

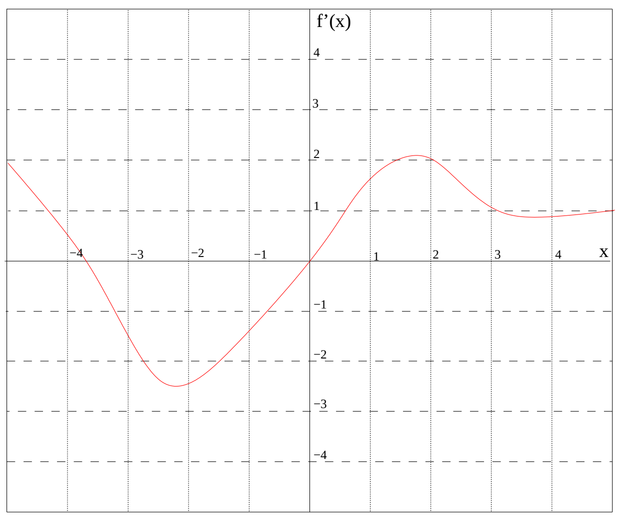
Since g is concave up, we know that $g'' > 0$. This means that g' is increasing, so $g'(4) > g'(2)$.

(c) What is the maximum possible value for $g(3)$? (Hint: try sketching a graph of g.) Explain your reasoning.

A sketch suggests the key idea: since g is concave up, the graph of g between $x = 1$ and $x = 5$ must be *lower* than the secant line connecting the points $(1, 9)$ and $(5, 3)$. This line passes through the point $(3, 6)$, and so it must be the case that $g(3) < 6$.

(8.) (10 points) The graph of $y = f(x)$ is given in the figure below. On the second set of axes below, sketch the graph of the derivative of f. Use the scale on the graph of f to help estimate values for f'.





(9.) (14 points) Suppose you decide to weave baskets and sell them for a living. Let $b = f(t)$ be the number of baskets you can weave in t hours, and let $d = g(b)$ be the number of dollars you can get for b baskets.

(a) Let $h(t) = g(f(t))$. Describe the function h in words.

The expression $h(t)$ is the number of dollars you can get after t hours of work weaving baskets.

(b) What are the units of $h'(t)$?

The units of $h'(t)$ are dollars per hour.

(c) Describe $f^{-1}(10)$ in words.

The expression $f^{-1}(10)$ represents the number of hours it takes you to weave 10 baskets.

(d) What are the units of $(f^{-1})'(b)$?

The units of $(f^{-1})'(b)$ are hours per basket.

(e) With any luck, you'll get better at basket-weaving as time passes – it will take you less time to weave each basket. State this in terms of the concavity of f. Explain your reasoning.

Since it takes you less and less time to weave each basket, you are weaving a greater and greater number of baskets per hour. This means that f' is increasing, so f is concave up.

(10.) (12 points) When you weigh yourself by standing on a bathroom scale, you push down on a spring inside the scale. As the spring compresses – that is, as it decreases in length – your body is acted on by two forces:

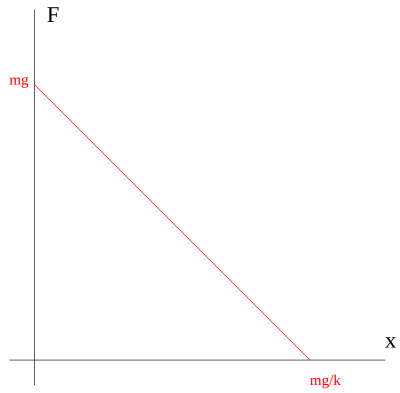
- Gravity exerts a downward force F_g on your body. The magnitude of this force is mg, where m is the mass of your body, and g is a constant.
- The spring in the scale exerts an upward force F_s on your body. The magnitude of this force is directly proportional to the total change in the spring's length. The constant of proportionality k is called the *spring constant* of the spring.

The net downward force F on your body equals the difference $F_g - F_s$.

(a) Write an expression for F as a function of x, the length by which the spring has been compressed.

$$F = mg - kx$$

(b) On the axes below, sketch a graph of F as a function of x, clearly labelling both intercepts.



(c) What is the significance of the x-intercept of this graph? Hint: we will refer to this x-value as x_{eq}.

When $x = x_{\text{eq}}$, the net force F is zero, because the gravitational force and the force from the spring are in *equilibrium*. Since there is no net force on your body, you stop sinking down into the scale.

(d) The mechanism inside the scale doesn't actually measure your mass m directly; instead, it measures the value of x_{eq}. However, it turns out that m and x_{eq} only differ by multiplication by a constant factor – that is, $m = c \cdot x_{\text{eq}}$, for some c. This means that the numbering of the scale's display can be chosen so that the scale gives a readout of your mass, after all.

What is the value of the constant c?

$$x_{\text{eq}} = \tfrac{mg}{k}, m = \left(\tfrac{k}{g}\right) x_{\text{eq}}$$

$$\boxed{c = \tfrac{k}{g}}$$