MATH 115 — SECOND MIDTERM EXAM

November 12, 2003

NAME: ____

INSTRUCTOR: ____________________ SECTION NO: ______________

1. Do not open this exam until you are told to begin.
2. This exam has 9 pages including this cover. There are 8 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed 2 sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones.

PROBLEM	POINTS	SCORE
1	16	
2	9	
3	16	
4	12	
5	12	
6	11	
7	10	
8	14	
TOTAL	100	

(1.) (16 points) Indicate whether each statement is true or false. Circle TRUE only if the statement is *always* true.

(a) If $x = 4$ is a critical point of the function f, then $f'(4) = 0$.

TRUE FALSE

(b) If $g'(x) < 0$ for $x < 3$, $g'(x) > 0$ for $x > 3$, and $g'(3) = 0$, then g has a local minimum at $x = 3$.

TRUE FALSE

(c) If $f'(x)$ is defined for all x, then $f(x)$ is defined for all x.

TRUE FALSE

(d) It is possible to have a local minimum of f at $x = c$ if $f''(c) = 0$.

TRUE FALSE

(e) If $f'(3) = 6.4$ and $g'(3) = 2.3$, then the graph of $f(x) - g(x)$ has a slope of 4.1 at $x = 3$.

TRUE FALSE

(f) If $f(x)$ is increasing for all x, then $f'(x)$ is increasing.

TRUE FALSE

(g) For a revenue function, R, and a cost function, C, if $R(q_0) > C(q_0)$ and $MR < MC$ at $q = q_0$, a company would be advised to increase q.

TRUE FALSE

(h) The profit function is always maximized if marginal revenue equals marginal cost.

TRUE FALSE

(2.) (9 points) Suppose you are given the following data about a differentiable function f:

- $f(3) = 7$
- $f'(3) = -4$.

(a) Find the local linearization of f near $x = 3$.

$$f(x) \approx f(3) + f'(3)(x-3)$$
$$= 7 - 4(x-3)$$

(b) Use linear approximation to estimate $f(3.1)$.

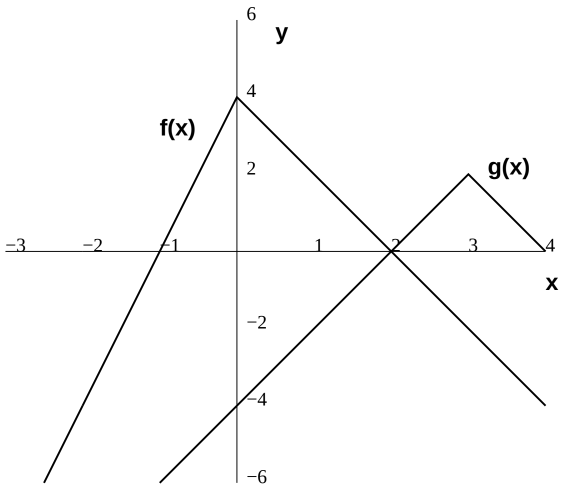
$$f(3.1) \approx 7 - 4(3.1-3)$$
$$= 7 - 4(0.1) = 6.6$$

(c) If $f''(3) < 0$, do you expect your approximation to be an overestimate or underestimate for $f(3.1)$? Explain, *using a sketch* to support your answer. Include all relevant features of the function on your sketch–and express your answer in a sentence.

We expect our linear approximation to be an overestimate: since the graph is concave down near $x = 3$, the tangent line there is above the graph near $x = 3$.

[Note: A sketch should also be included here.]

(3.) (16 points) The graphs of two functions f and g are shown below. [Note that the scales on the axes are not the same.]



(a) If $h(x) = f(g(x))$, compute $h'(1)$.

$$h'(1) = f'(g(1)) \cdot g'(1) = f'(-2) \cdot g'(1) = 4 \cdot 2 = 8$$

(b) If $k(x) = f(x) \cdot g(x)$, compute $k'(1)$.

$$k'(1) = f'(1) \cdot g(1) + f(1) \cdot g'(1) = -2 \cdot (-2) + 2 \cdot 2 = 8$$

(c) If $q(x) = \frac{f(x)}{g(x)}$, compute $q'(1)$.

$$q'(1) = \frac{g(1) \cdot f'(1) - f(1) \cdot g'(1)}{g^2(1)} = \frac{-2 \cdot (-2) - 2 \cdot (2)}{(-2)^2} = 0$$

(d) If $t(x) = \ln(g(x))$, compute $t'(1)$.

$$t'(1) = \frac{1}{g(1)} \cdot g'(1) = \frac{1}{-2} \cdot 2 = -1$$

Note: ln(g(x)) is only defined where g(x) is positive.
Therefore t (and hence t') are undefined at x=1.
This was an oversight of the test writer, but was caught during grading and graded correctly.

(4.) (12 points) Consider the function:

$$f(x) = e^{\frac{-(ax)^2}{2}}, \qquad \text{for } a \text{ a positive constant.}$$

The graph of $y = f(x)$ is the (in)famous "bell curve," which occurs frequently in statistics, and occasionally in heated political debates as well.

(a) Compute $f''(x)$. Show your work.

Use the chain rule:

$$f'(x) = e^{\frac{-(ax)^2}{2}} \cdot (-(ax)) \cdot a = (-a^2x) \cdot e^{\frac{-(ax)^2}{2}}$$

Now use the product rule, together with the previous line:

$$f''(x) = -a^2 \cdot e^{\frac{-(ax)^2}{2}} + (-a^2x) \cdot \left[(-a^2x) \cdot e^{\frac{-(ax)^2}{2}}\right]$$

$$f''(x) = a^2 e^{\frac{-(ax)^2}{2}} (a^2x^2 - 1)$$

(b) For which value of a does the function f have an inflection point at $x = 3$?

First let's find out where $f''(x) = 0$, since this is a prerequisite for an inflection point. Since $e^k > 0$ for any k and $a^2 \neq 0$, we need only find out where $a^2x^2 - 1 = 0$. This happens when $x = \pm 1/a$. If $x = 3$ and a is positive, we must have $a = 1/3$. To assure that this is an inflection point of f, we can check the sign of $f''(x)$ to the left and right of $x = 3$. We see that $f''(x)$ is negative to the left of $x = 3$ and positive to the right of $x = 3$. Thus, the function changes from concave down to concave up at $x = 3$ when $a = 1/3$.

(5.) (12 points) Suppose p is a cubic polynomial function. Recall that this means that

$$p(x) = a_3x^3 + a_2x^2 + a_1x + a_0,$$

for some constants a_0, a_1, a_2, a_3, with $a_3 \neq 0$.

(a) If $p(0) = 1$, then what is the value of a_0?

$$a_0 = p(0) = 1$$

(b) If $p'(0) = 1$, then what is the value of a_1?

$$a_1 = p'(0) = 1$$

(c) If $p''(0) = 1$, then what is the value of a_2?

$$p''(0) = 2a_2,$$
$$\text{so } 2a_2 = 1, \text{ and } a_2 = \tfrac{1}{2}$$

(d) If $p'''(0) = 1$, then what is the value of a_3?

$$p'''(0) = 6a_3$$
$$\text{so } 6a_3 = 1, \text{ and } a_3 = \tfrac{1}{6}$$

(e) Find the formula for a cubic polynomial function q that satisfies:

$$q(0) = 2, \quad q'(0) = -1, \quad q''(0) = 5, \quad q'''(0) = 4.$$

[Note: You may use the information that you found in parts (a)-(d) to help you.]

$$q(x) = \tfrac{4}{6}x^3 + \tfrac{5}{2}x^2 - x + 2$$

(6.) (11 points) The equation $x^2 - xy + y^2 = 3$ represents a "rotated ellipse"—that is, an ellipse whose axes are not parallel to the coordinate axes.

(a) Find the points at which this ellipse crosses the x-axis.

Plug in $y = 0$ and solve for x:

$$x^2 = 3$$

$$x = \sqrt{3}, -\sqrt{3}$$

(b) Show that the tangent lines at these points are parallel.

Find $\frac{dy}{dx}$, using implicit differentiation:

$$2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{y-2x}{2y-x}$$

When $y = 0$, $x = \pm\sqrt{3}$, we have $\frac{dy}{dx} = 2$.

Since $\frac{dy}{dx}$ is the slope of the tangent line, we see that both lines have slope 2, so they are parallel.

(c) Under what conditions on x (if any) would a tangent to the curve be vertical? Explain.

The tangent line is vertical when the denominator of $\frac{dy}{dx}$ is zero (and the numerator is not zero). This happens when $2y = x$ (and $y \neq 2x$).

(7.) (10 points) For some positive constant C, a patient's temperature change, T, due to a dose, D, of a drug is given by

$$T = f(D) = \left(\frac{C}{2} - \frac{D}{3}\right) D^2$$

(a) What dosage maximizes the temperature change?

$f'(D) = CD - D^2$, $f''(D) = C - 2D$

Critical point when $f'(D) = 0$, that is, when $D = 0$ or $D = C$.

Since $f''(C) = -C < 0$, the function $f(D)$ has a maximum when $D = C$.

Note that since $D = C$ is the *only* critical point for $D > 0$, and it is a local maximum, it must also be a global maximum for $D > 0$.

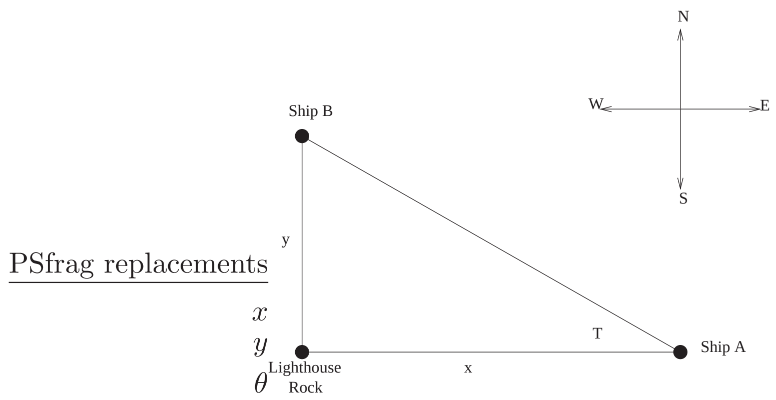
(Note that $f''(0) = C > 0$, so $D = 0$ is a minimum.)

(b) The sensitivity of the body to the drug is defined as dT/dD. What dosage maximizes sensitivity?

Critical point when $(\frac{dT}{dD})' = f''(D) = 0$.

This happens when $D = C/2$. Check that $(\frac{dT}{dD})''(C/2) = f'''(C/2) = -2 < 0$, so this is indeed a maximum, and since this is the only critical point, $D = \frac{C}{2}$ is a global maximum.

(8.) (14 points) Ship A is travelling due west toward Lighthouse Rock at a speed of 15 kilometers per hour (km/hr). Ship B is travelling due north away from Lighthouse Rock at a speed of 10 km/hr. Let x be the distance between Ship A and Lighthouse Rock at time t, and let y be the distance between Ship B and Lighthouse Rock at time t, as shown in the figure below.



(a) Find the distance between Ship A and Ship B when $x = 4$ km and $y = 3$ km.

$$\text{distance} = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = 5 \text{ km}$$

(b) Find the rate of change of the distance between the two ships when $x = 4$ km and $y = 3$ km.

$D^2 = x^2 + y^2$ where D, x, and y are all functions of t.

Thus, $2D(\frac{dD}{dt}) = 2x(\frac{dx}{dt}) + 2y(\frac{dy}{dt})$

$\frac{dx}{dt} = -15\frac{km}{hr}$, and $\frac{dy}{dt} = 10\frac{km}{hr}$, so:

$$\frac{dD}{dt} = \frac{x(\frac{dy}{dt}) + y(\frac{dx}{dt})}{D}$$

When $x = 4$, $y = 3$, we have:

$$\frac{dD}{dt} = \frac{(4)(-15)+(3)(10)}{5} = -6\frac{km}{hr}$$

(c) Let θ be the angle shown in the figure. Find the rate of change of θ when $x = 4$ km and $y = 3$ km.

Note that $\tan(\theta) = \frac{y}{x}$.

Thus, $\frac{1}{\cos^2(\theta)}(\frac{d\theta}{dt}) = \frac{x(\frac{dy}{dt}) - y(\frac{dx}{dt})}{x^2}$,

$$\frac{d\theta}{dt} = \frac{4(10) - 3(-15)}{16}(\frac{16}{25})$$

$$= \frac{85}{16}(\frac{16}{25}) = \frac{85}{25} = \frac{17}{5}\frac{radians}{hour}.$$