MATH 115 — FINAL EXAM

December 15, 2003

NAME: SOLUTION KEY

INSTRUCTOR: ____________________ SECTION NO: ____________________

1. Do not open this exam until you are told to begin.
2. This exam has 11 pages including this cover. There are 11 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed 2 sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones.

PROBLEM	POINTS	SCORE
1	15	
2	12	
3	9	
4	9	
5	5	
6	8	
7	8	
8	10	
9	8	
10	11	
11	5	
TOTAL	100	

(1.) (15 points) Let g be a differentiable function. Find formulas for the derivatives of the each of the following. [Your derivative formulas may contain g and/or g'.]

(a) $m(x) = \sin(x) \cdot g(x)$

$m'(x) = \cos(x) \cdot g(x) + \sin(x) \cdot g'(x)$

(b) $t(x) = \frac{\sin(x)}{g(x)}$

$t'(x) = \frac{g(x) \cdot \cos(x) - \sin(x) \cdot g'(x)}{g^2(x)}$

(c) $p(x) = \sin(a \cdot g(x))$, where a is a constant

$p'(x) = \cos(a \cdot g(x)) \cdot a \cdot g'(x)$

(d) $k(x) = \sin^2(g(x))$

$k'(x) = 2\sin(g(x)) \cdot \cos(g(x)) \cdot g'(x)$

(e) $f(x) = \sin(g(x^2))$

$f'(x) = \cos(g(x^2)) \cdot g'(x^2) \cdot 2x$

(2.) (12 points) Given the following:

- f is an ***even*** function such that $\int_0^1 f(x)\,dx = 5$,
- g is an ***odd*** function such that $\int_0^1 g(x)\,dx = 7$.

Compute the following definite integrals. If you do not have enough information for a given computation, write "not enough information."

(a) $\int_0^1 (f(x) - g(x))\,dx = -2$

(b) $\int_0^1 3g(x)\,dx = 21$

(c) $\int_0^1 f(x) \cdot g(x)\,dx =$ not enough information

(d) $\int_3^4 f(x-3)\,dx = 5$

(e) $\int_{-1}^1 (f(x) + g(x))\,dx = 10$

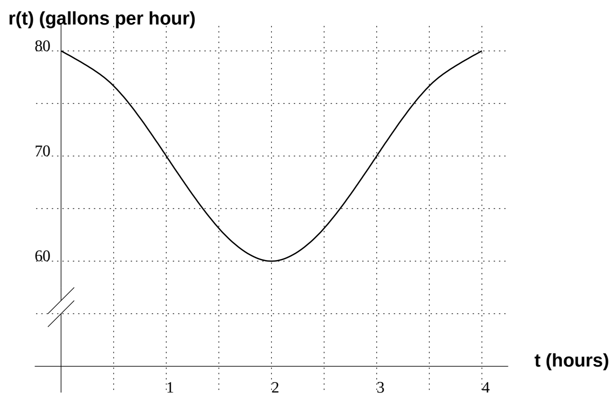
(f) $\int_0^1 f(g(x))\,dx =$ not enough information

(3.) (9 points) A large tank is being filled with water. The flow rate of water into the tank, in units of gallons per hour, is given by

$$r(t) = 70 + 10\cos\left(\frac{\pi t}{2}\right),$$

where t is measured in hours.

(a) Sketch an accurate graph of $r(t)$ on the following axes.



(b) Use a definite integral to express the area under the graph of $r(t)$ between the vertical lines $t = 0$ and $t = 3$.

$$\int_0^3 r(t)\,dt$$

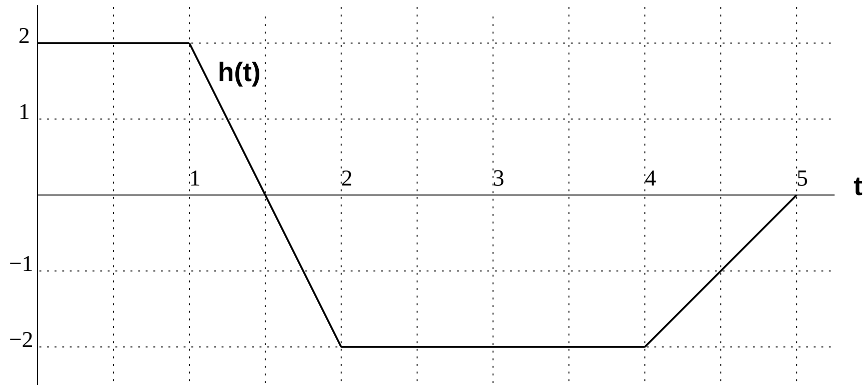
(c) What is the practical meaning of integral in part (b)? Be sure to include units in your answer.

The integral represents the total number of gallons of water that flowed into the tank during the three hours from $t = 0$ to $t = 3$. This is *not* necessarily the same as the total amount of water in the tank at $t = 3$ hours, because the tank may not have been empty at $t = 0$ hours.

(d) Give an expression for the average flow rate between $t = 0$ and $t = 3$? Do not estimate—i.e., leave your answer as a formula.

$$\frac{1}{3}\int_0^3 r(t)\,dt$$

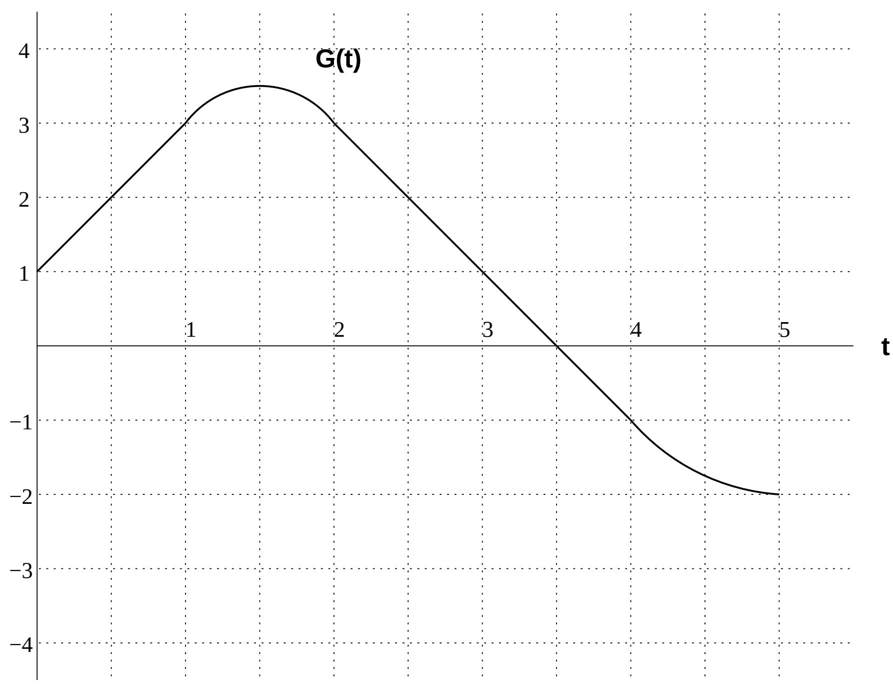
(4.) (9 points) The following is the graph of a function $h(t)$:



(a) If $H(t)$ is a function such that $H'(t) = h(t)$, complete the following table:

t	0	1	2	3	4	5
$H(t)$	3	5	5	3	1	0

(b) Let G be another function whose derivative equals $h(t)$ (i.e., $G'(t) = h(t)$). On the axes below, sketch the graph of G, given that the graph passes through the point $(1, 3)$.



(5.) (5 points) Let $f(x) = 1/x$. Use the *limit definition* of the derivative (and some algebra) to compute $f'(x)$. [Show **all** work.]

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} \\ &= \lim_{h\to 0} \frac{\frac{x-(x+h)}{x^2+xh}}{h} \\ &= \lim_{h\to 0} \frac{\frac{-h}{x^2+xh}}{h} \\ &= \lim_{h\to 0} \frac{-1}{x^2+xh} \\ &= \frac{-1}{x^2} \end{aligned}$$

(6.) (8 points)

(a) Given $F(x) = x\ln(x) - x + C$, show that $F'(x) = \ln(x)$. [Show all your work.]

$$\begin{aligned} F'(x) &= \ln(x) + x \cdot \frac{1}{x} - 1 + 0 \\ &= \ln(x) + 1 - 1 \\ &= \ln(x) \end{aligned}$$

(b) If $F(1) = 3$, find C.

$$\begin{aligned} F(1) &= 1 \cdot \ln(1) - 1 + C \\ &= 1 \cdot 0 - 1 + C \\ &= -1 + C \\ &= 3, \end{aligned}$$

so $C = 4$.

(c) Evaluate $\int_1^3 \ln(x)\,dx$. [Give and *exact* answer, not an approximation.]

Since $F'(x) = \ln(x)$, the fundamental theorem of calculus says that:

$$\begin{aligned} \int_1^3 \ln(x)\,dx &= F(3) - F(1) \\ &= (3\cdot\ln(3) - 3 + C) - (1\cdot\ln(1) - 1 + C) \\ &= 3\cdot\ln(3) - 2. \end{aligned}$$

(7.) (8 points) The following is a table of values of a continuous function f:

x	0	20	40	60	80	100
$f(x)$	1.2	2.8	4.0	4.7	5.1	5.2

(a) Use a left-hand sum with five intervals to estimate the definite integral $\int_0^{100} f(x)\,dx$. Show your work.

We use five rectangles to estimate the area. Each rectangle has width 20, and height given by the value of the function at the left endpoint. Thus we have:

$$\int_0^{100} f(x)\,dx \approx 20 \cdot (1.2 + 2.8 + 4.0 + 4.7 + 5.1) = 356.$$

(b) Assuming that f is monotonic (i.e., always increasing or decreasing on the interval), how many intervals must you use to guarantee that the left hand sum is within .1 of the actual value of the integral?

Since f is monotonic, the actual value of the integral is between the value of any left hand sum and any right hand sum. If we use n intervals for both the left hand sum and right hand sum, then we have the formula:

$$\begin{aligned} RHS - LHS &= (100/n) \cdot (5.2 - 1.2) \\ &= (400/n). \end{aligned}$$

If we can get this value to be less than .1, then we will guarantee that the left hand sum is within .1 of the true value of the integral. So we need:

$$\begin{aligned} (400/n) &< .1 \\ 400 &< .1n \\ 4000 &< n. \end{aligned}$$

(c) Given the information you have, is your left-hand sum an underestimate or an overestimate? Explain.

Because f is monotonically increasing, the value of any left-hand sum will be an underestimate of the actual value of the integral.

(8.) (10 points) Saruman the White is creating an army of orcs to cut down all the trees in Fangorn Forest. Saruman is currently trying to decide exactly how large the army should be in order to destroy the forest as quickly as possible.

The trouble is, orcs aren't very efficient. In very small armies they tend to work pretty well — one orc will emerge as the leader, and he will have good control over the others. They also organize fairly well in very large armies, once a military structure is established. In medium-sized armies, though, the orcs spend a lot of time fighting for dominance, and as a result they can't work very efficiently.

Saruman has noticed this, of course. His research indicates that an army of x *thousand* orcs, will be able to cut down

$$T(x) = \frac{x^3}{3} - 3x^2 + 8x \qquad \text{thousand trees per hour.}$$

(a) If Saruman is capable of producing an army of up to 3000 orcs, how many should he produce in order to maximize the hourly destruction of trees? [Saruman does not have a graphing calculator and must be convinced by the methods of calculus.]

We optimize $T(x)$:

$$\begin{aligned} T'(x) &= x^2 - 6x + 8 \\ &= (x-2)(x-4) \end{aligned}$$

So T has critical points at $x = 2$ and $x = 4$. We have that $T''(x) = 2x - 6$, and so we see that $x = 2$ is a local maximum, and $x = 4$ is a local minimum. We note that

$$T(2) = \frac{8}{3} - 12 + 16 = \frac{20}{3},$$

but we must also check endpoints: $T(0) = 0$, and $T(3) = 6$. We see that the maximum is at $x = 2$, so Saruman should produce an army of 2000 orcs.

(b) Does your answer change if Saruman can produce up to 4000 orcs? If so, how many should he produce now?

No, the answer does not change. $T(4)$ is a local minimum, so it must be less than $T(2)$. Since there are no other critical points between the old endpoint $x = 3$ and the new one, there is nothing else to check.

(c) Does your answer change if Saruman can produce up to 6000 orcs? If so, how many should he produce now?

The value of T at the new endpoint is

$$\begin{aligned} T(6) &= \frac{216}{3} - 108 + 48 \\ &= \frac{216}{3} - 60 \\ &= \frac{216}{3} - \frac{180}{3} \\ &= \frac{36}{3} \\ &= 12. \end{aligned}$$

This is larger than $T(2)$, so Saruman should now produce an army of 6000 orcs.

(9.) (8 points) Winter is here! Soon we will have icicles. Consider an icicle in the shape of a right circular cone. The sun is causing the icicle to lengthen. As its length, h, is increasing at the rate of 0.5 cm/hr, the radius, r, of the cone is decreasing at the rate of 0.02 cm/hr. When the icicle is 12 cm long and its radius is 1 cm, is the volume of the icicle increasing or decreasing? At what rate is the volume changing? [The volume of a right circular cone is given by $V = \frac{1}{3}\pi r^2 h$. Note that in this problem, both h and r are functions of time.]

Take the derivative of V with respect to time:

$$\begin{aligned} \frac{dV}{dt} &= \frac{\pi}{3}\left(2r \cdot \frac{dr}{dt} \cdot h + r^2 \cdot \frac{dh}{dt}\right) \\ &= \frac{\pi}{3}\left(2 \cdot 1 \cdot (-.02) \cdot 12 + 1^2 \cdot .5\right) \\ &= \frac{\pi}{3}(-.48 + .5) \\ &= \frac{\pi}{3} \cdot (.02) \approx 0.0209 \ \text{cm}^3/\text{hr}. \end{aligned}$$

This number is positive, so the volume is increasing at the rate of approximately 0.02 cm^3/hr.

(10.) (11 points) You are designing a cylindrical bucket. The bucket must have a bottom, but it will have no lid, and you have 1000 square inches of steel sheet to use for the bucket.

If the radius of the bucket is r and the height is h, for what values of r and h does the bucket have maximum possible volume? What is this maximum volume? Show all your work, and clearly indicate your final answers below.

Let V be the volume of the bucket, and let S be the surface area of the bucket. We have that

$$V = \pi \cdot r^2 \cdot h, \quad S = \pi \cdot 2r \cdot h + \pi \cdot r^2.$$

Since we have 1000 square inches of sheet to work with, we have the equation

$$1000 = \pi \cdot 2r \cdot h + \pi \cdot r^2,$$

which we can solve for h:

$$h = \frac{1000 - \pi \cdot r^2}{\pi \cdot 2r}.$$

Now we can write V as a function of r only:

$$\begin{aligned} V(r) &= \pi \cdot r^2 \cdot \left(\frac{1000 - \pi \cdot r^2}{\pi \cdot 2r}\right) \\ &= 500r - \frac{\pi}{2}r^3 \end{aligned}$$

Now we optimize this function:

$$V'(r) = 500 - \frac{3\pi}{2}r^2 = 0,$$

so V has critical points at $r = \pm\sqrt{\frac{1000}{3\pi}}$. We restrict our attention to the positive critical point. Since V'' is negative at this critical point, we find that this positive value of r is a local maximum. Since we are only interested in values of r that are nonnegative, and since $V(0) = 0$, we conclude that $V(\sqrt{\frac{1000}{3\pi}})$ is the global maximum we seek. The final answers are obtained by plugging in $r = \sqrt{\frac{1000}{3\pi}}$:

$$h = \frac{1000 - \frac{1000}{3}}{\pi \cdot 2\sqrt{\frac{1000}{3\pi}}} = \frac{\frac{1000}{3}}{\sqrt{\frac{1000\pi}{3}}} = \frac{1000}{3} \cdot \frac{\sqrt{3}}{\sqrt{1000\pi}} = \sqrt{\frac{1000}{3\pi}}.$$

Thus, the volume is

$$V = \pi \left(\frac{1000}{3\pi}\right)^{\frac{3}{2}}$$

Optimal value of $r = \sqrt{\frac{1000}{3\pi}}$ inches

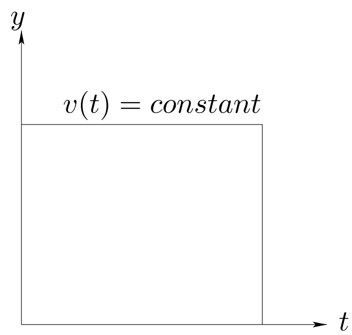
Optimal value of $h = \sqrt{\frac{1000}{3\pi}}$ inches

Maximum volume $= \pi \left(\frac{1000}{3\pi}\right)^{\frac{3}{2}}$ cubic inches

(11.) (5 points) Suppose that on your visit home over break you meet a friend who is now taking precalculus at your old high school. He knows the formula "distance travelled = rate × time." He also knows some students who are taking the calculus course at the high school, and he has heard there is a more general formula, "distance travelled = area under the velocity curve," that computes the distance, even when the velocity is not constant. He asked those students to explain this second formula, but they just shrugged and said he would have to wait until he learned calculus to get an explanation.

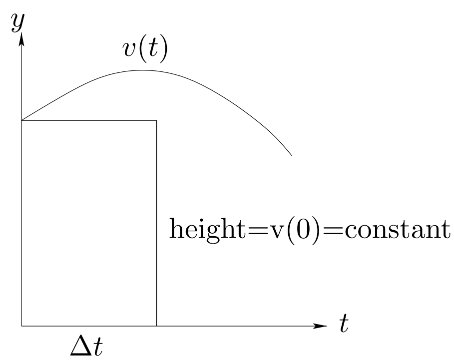
Write down what you would tell your friend to explain why the second formula holds and how it is related to the formula he has learned in precalculus. Be sure to include any appropriately labelled graphs you might draw in making your explanation.

The precalculus formula of $d = v\,t$ gives the formula relating the distance, time and velocity when the velocity is constant over time. One can think of this as the area under a velocity curve given by a horizontal line, as can be seen in (Fig. 1). However, when the velocity function varies



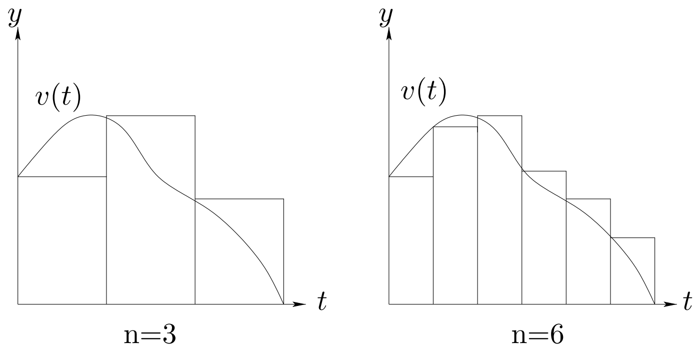
$Fig.1$

with time, we can approximate it on small intervals to be a constant function. The height of the rectangle gives an approximation to the actual velocity function. Therefore the actual distance travelled over this small interval is approximately the distance travelled by the constant velocity we are approximating the actual velocity by. So the actual distance is approximately the height of the rectangle × the time interval we are using. Note that this is actually the area of the rectangle! See (Fig. 2).



$Fig.2$

Now to get an approximation of the total distance travelled, one merely adds up the various rectangles used in the approximation. As the time interval used decreases to 0, the number of rectangles used increases to infinity and we see that the area of the rectangles actually fills out the entire area under the velocity curve. See (Fig. 3) for the cases of 3 rectangles and 6 rectangles. This is how one goes from the simple precalculus formula of distance = rate $\times$ time to the area



$Fig.3$

under the velocity curve being the distance travelled.

Please **print** your name here: Name ____________________