MATH 115 — FIRST MIDTERM EXAM

October 13, 2004

NAME: SOLUTION KEY

INSTRUCTOR: ____________________ SECTION NO: __________

1. Do not open this exam until you are told to begin.
2. This exam has 10 pages including this cover. There are 10 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed 2 sides of a 3 by 5 note card.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones and remove all headphones.

PROBLEM	POINTS	SCORE
1	12	
2	9	
3	6	
4	4	
5	9	
6	11	
7	10	
8	14	
9	12	
10	13	
TOTAL	100	

Midterm 1 Solutions

1. (2 points each) Circle "True" or "False" for each of the following problems. Circle "True" only if the statement is *always* true. No explanation is necessary.

(a) $\log^{-1}(x) = \frac{1}{e^x}$.

True False

(b) If a function is continuous at a point a, then it must also be differentiable at a.

True False

(c) Suppose f is a continuous function on the interval $[5, 8]$ and that $f(5) = -2$ and $f(8) = 3$. Then f has a zero on the interval $(5, 8)$.

True False

(d) $\lim_{x \to 6} \frac{|x-7|}{x-7}$ exists and is equal to -1.

True False

(e) Suppose f is a continuous function and f is concave up on the interval $(-10, 10)$. If $f'(1) = -2$, it is possible that $f'(4) = -3$.

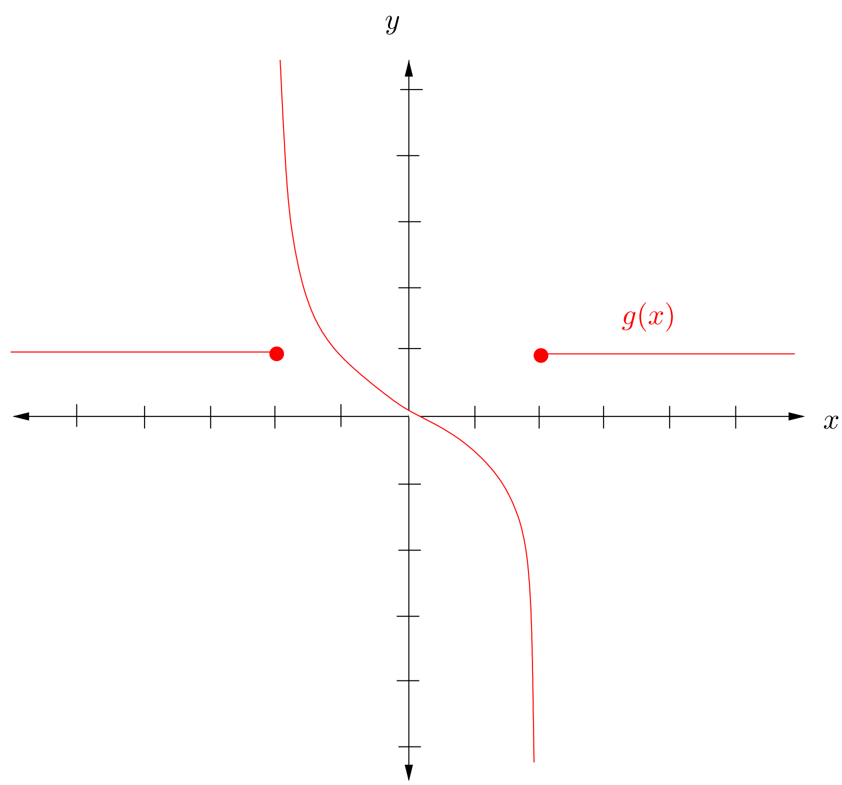
True False

(f) Suppose f is a continuous function, $f(1) = 6$, and $f'(x) > 0$ for all x between 0 and 5. Then it is possible that $f(4) = 6$.

True False

2.(9 points) On the axes below, sketch a graph of a single **function**, g, with **all** of the following properties.

- $g(-2) = g(2) = 1$
- $g'(x) = 0$ for $x < -2$ and $x > 2$
- $g'(x) < 0$ for $-2 < x < 2$
- $\lim_{x \to -2^+} g(x) = \infty$ and $\lim_{x \to 2^-} g(x) = -\infty$
- $g''(x) > 0$ for $-2 < x < 0$
- $g''(x) < 0$ for $0 < x < 2$



3. (6 points) A group of researchers in Costa Rica is studying the number of resplendent quetzals (these are birds) that nest in Monteverde Cloud Forest Preserve each year. The function f gives the number of quetzals the researchers count in the park on day t. Write an expression involving f that models each of the situations (a)-(c) below.

(a) After determining $f(t)$, the researchers discover they forgot to include an area of the park that houses 50 quetzals year round. Write an expression representing the number of quetzals the researchers should use for their count.

$$\text{The appropriate number of quetzals} = f(t) + 50.$$

(b) The number of visitors to the park is a function of the number of quetzals in the park. Suppose the function is given by $g(q)$ where q is the number of quetzals in the park. Write an expression that gives the number of visitors to the park on day t, based on the information that the researchers have.

$$\text{The number of visitors to the park on day } t = g(f(t) + 50).$$

(c) The researchers discover another sloppy calculation. They had speculated that on day T they would count the maximum number of quetzals. They found that the maximum quetzal count actually took place in the park 5 days earlier than they had anticipated. Find a formula for the maximum count, M, in terms of day T.

$$M = f(T - 5) + 50$$

4. (4 points) The volume of a cylinder of radius r and height h is given by $V = \pi r^2 h$. If 6 times the height plus 2 times the radius must equal 36, determine a formula for the volume of the cylinder in terms of the radius.

The conditions given tell us that $6h + 2r = 36$. Since we want the formula for the volume of the cylinder in terms of the radius, we need to eliminate h. Therefore, we solve $6h + 2r = 36$ for h, obtaining $h = 6 - \frac{1}{3}r$. Now we plug this back into the given formula for V and get

$$V = \pi r^2 (6 - \frac{1}{3} r).$$

5. (9 points) The number of socks you own decreases according to the number of loads of laundry you've done since the beginning of the school year. After your first load of laundry you have 10 pairs of socks remaining and after 20 loads you are down to your last pair of socks. Find an exponential function that models this situation, and approximate the number of pairs of socks that you had when you started the semester. Interpret your approximation into a reasonable answer, and express your answer in a sentence. [Show your work!]

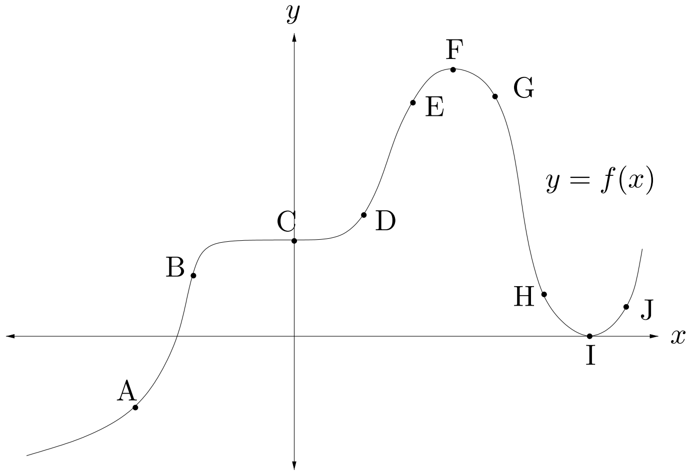
One can begin by noting that we are actually given that the number of socks decreases exponentially, not the number of pairs. However, since the number of socks is just twice the number of pairs, we also have that the number of pairs decreases exponentially.

Let P be the number of pairs of socks, P_0 the initial number of pairs you own, and n the number of loads of laundry you've done. Then we have the equation $P = P_0e^{kn}$ for a constant k. We are then given that $10 = P_0e^k$ and $1 = P_0e^{20k}$. Dividing these we obtain $\frac{1}{10} = e^{19k}$, which gives us that $k = -0.12$.

Then we can plug this back into $10 = P_0e^k$ to get that $P_0 = 11.28$. However, as we are talking about pairs of socks and one can't really have .28 pairs of socks, we conclude that you started with 11 pairs of socks.

[Note: This problem can also be worked with the equation in the form of $P = P_0a^n$.]

6. (11 points) The graph of a continuous differentiable function f is given below. Use the graph to answer the following. No explanation necessary.



(a) List all labelled points (if any) where f' and f'' are both positive.

A, D, J

(b) List all labelled points (if any) where f' and f'' are both negative.

G

(c) List all labelled points (if any) where f and f' are both positive.

B, D, E, J

(d) List all labelled points (if any) where f and f' are both both negative.

none

(e) List all labelled points (if any) where at least two of f, f', f'' are zero.

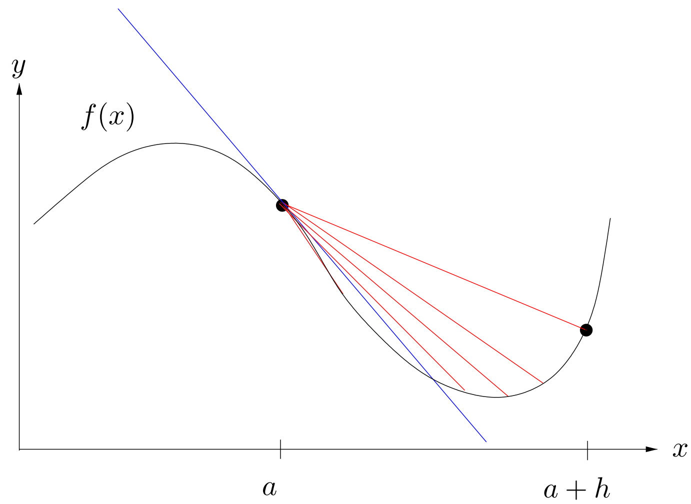
C, I

7. (10 points) For this problem f is differentiable everywhere.

(a) Write the limit definition of the derivative of the function f at the point a.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(b) On the graph below, show how the average rate of change of f between $x = a$ and $x = a + h$ is related to the derivative at the point a. Give a brief explanation of your illustration including how the limit as $h \to 0$ is demonstrated in your picture.

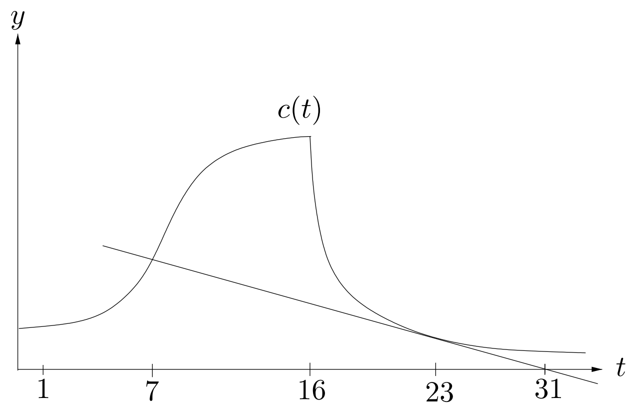


The average rate of change of f between $x = a$ and $x = a+h$ is the slope of the red line connecting $f(a)$ and $f(a+h)$. This is an approximation to the derivative of f at $x = a$, which is the slope of the tangent line (shown in blue) of f at $x = a$. One can see as the h gets smaller ($h \to 0$), the slopes of the red lines become better and better approximations to the slope of the tangent line to f and $x = a$.

(c) Write the limit definition for $f'(2)$ if $f(x) = e^{\sin 2x}$. [You do not need to find the limit or approximate $f'(2)$.]

$$f'(2) = \lim_{h \to 0} \frac{e^{\sin 2(2+h)} - e^{\sin 4}}{h}$$

8. (14 points) As Sweetest Day (October 16th) approaches, millions of Americans flock to stores to buy their special someone a card. The number of cards sold can be approximated by the continuous function c graphed below where $c(t)$ gives the number of cards sold on day t and $t = 1$ corresponds to October 1.



(a) Are there any t values where the function may not be differentiable? Explain.

The function c is not differentiable at $t = 16$ as the function has a sharp corner here (or a vertical tangent–either will do).

(b) Explain the concavity of the graph between October 1 and October 16 in the context of this problem.

As the days pass after October 1st, people are flocking to the stores at an increasing rate until approximately October 8th. So more cards are being sold each successive day between October 1st and the 8th. At this point, the number of people still needing to buy cards slows and the rate is decreasing until to approximately 0 on Sweetest day, when there are nearly the same number of cards sold as on the day before.

(c) If on October 1 there are 30,000 cards sold and on October 23 there are 25,000 cards sold, what is the average rate of change of $c(t)$ over this time? Express your final answer in sentence form and in the context of this problem.

The average rate of change of $c(t)$ over the interval $t = 1$ to $t = 23$ is given by the expression

$$\frac{c(23) = c(1)}{23 - 1} = \frac{25,000 - 30,000}{22} = \frac{-5,000}{22} \approx -227.$$

This means that between October 1st and October 23 the number of cards decreases by approximately 227 cards each day.

(d) How many cards were sold on October 7? Show your work.

One needs to use the points $(23, 25,000)$ and $(31, 0)$ to find the equation of the pictured line. The slope of the line is $m = \frac{0-25,000}{31-23} = -3125$. So the equation of the line is $y = -3125(t - 31)$. To find how many cards were sold on October 7th we only need to plug $t = 7$ in, which gives 75,000. Therefore, on October 7th there were 75,000 cards sold.

9. (12 points) As fall progresses the trees in the Arboretum gradually change color. The function f gives the percentage of leaves on a particular tree that have began to change colors as a function of the number of days since September 30. (October 1st corresponds to $t = 1$.) All answers should be in complete sentences.

(a) Give a practical interpretation in everyday terms describing what $f(10) = 15$ means in the context of this problem.

On October 10th 15% of the leaves have began to change color.

(b) Give a practical interpretation for what $f'(15) = 9$ means in the context of this problem.

On October 15th approximately 9% more of the tree's leaves will begin to change color during the day.

(c) Give a practical interpretation for what $f^{-1}(3) = 6$ means in the context of this problem.

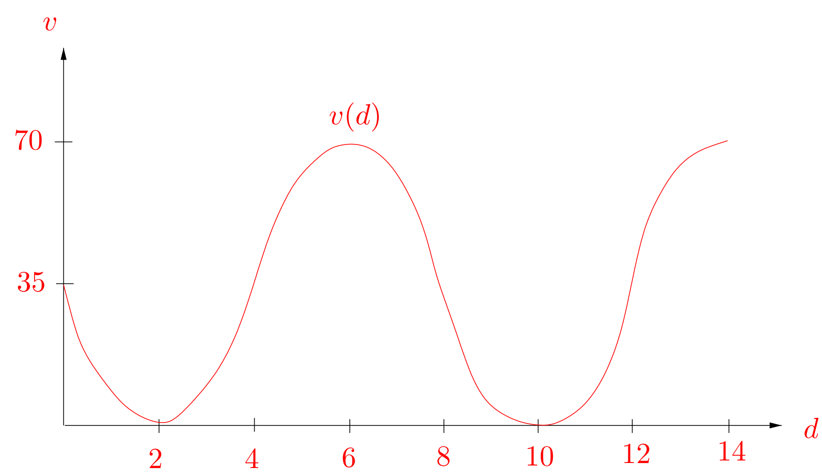
When 3% of the leaves have began to change colors it is October 6th.

(d) Give a practical interpretation describing what $(f^{-1})'(40) = 0.5$ means in the context of this problem.

When 40% of the tree's leaves have begun to change colors, it will take approximately 12 hours for the next percent to change colors.

10. (13 points) The traffic on US-23 between Brighton and Ann Arbor is stop and go every weekday morning. I merge onto US-23 South at Brighton travelling 35 miles per hour. The traffic is bad and I must immediately slow down, finally coming to a stop 2 miles after I got on the highway. I am able to speed right up again, and I reach my maximum speed of 70 miles per hour six miles after I merged onto US-23. There are again traffic problems and I must slow again, coming to a stop 4 miles after I reached my peak speed. Suppose my speed continues in the same pattern until I reach the Ann Arbor exit, 13 miles after I merged onto the highway at Brighton.

(a) Assume that my speed may be modelled by a trigonometric function and sketch a graph of my speed as I travel south on US-23. Let the horizontal axis represent my distance from the Brighton entrance to the highway. Be sure to appropriately label the axes.



(b) Determine a trigonometric function, v, giving my speed as a function of d, my distance from Brighton.

$$v(d) = -35\sin(\frac{\pi}{4}d) + 35$$

(c) What was my speed when I reached Ann Arbor?

To find the speed as I reach Ann Arbor, one merely needs to plug $d = 13$ into $v(d)$, obtaining 59.75 miles per hour.

(d) What are the units of v'?

First note that v is a function of d, not time which makes this a little trickier. So we are really looking at $v'(d) = \frac{dv}{dd}$. Then one sees that the units are (miles/hour)/miles = 1/hour.