November 17, 2004

NAME: SOLUTION KEY

INSTRUCTOR: ______________________ SECTION NO: __________

1. Do not open this exam until you are told to begin.
2. This exam has 9 pages including this cover. There are 9 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed 2 sides of a 3 by 5 note card.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones and all iPods and other sound devices.

PROBLEM	POINTS	SCORE
1	15	
2	14	
3	10	
4	6	
5	8	
6	13	
7	10	
8	12	
9	12	
TOTAL	100	

1. (3+4+4+4 points) Suppose that f and g are differentiable functions with values given by the following table:

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	2	5	-1	-6
4	4	2	12	-2

(a) Find the derivative of $n(x) = \pi^\pi + e^{\log 15} + f(2)$ when $x = 4$.

$$n'(x) = 0$$

(b) Find $h'(2)$ if $h(x) = \frac{\ln(f(x))}{g(x)}$.

$$h'(x) = \frac{\frac{1}{f(x)}f'(x)g(x) - \ln(f(x))g'(x)}{[g(x)]^2}$$

$$h'(2) = \frac{6\ln 2 - \frac{5}{2}}{25}$$

(c) Find the derivative of $k(x) = f(x)\cos\left(\frac{\pi}{6}x\right)$ when $x = 2$.

$$k'(x) = f'(x)\cos\left(\frac{\pi}{6}x\right) - f(x)\frac{\pi}{6}\sin\left(\frac{\pi}{6}x\right)$$

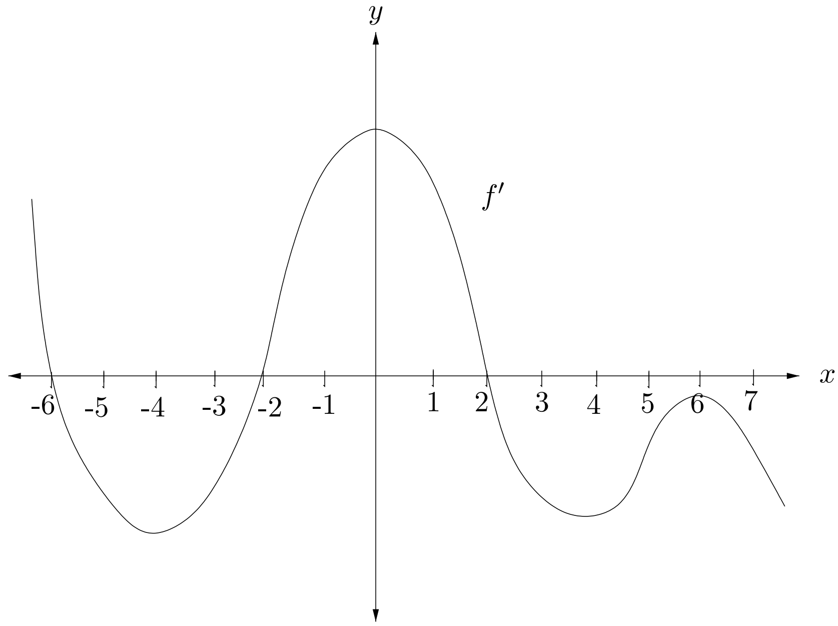
$$k'(2) = -\frac{1}{2} - \frac{\pi}{3}\cdot\frac{\sqrt{3}}{2}$$

(d) Find $j'(2)$ if $j(x) = f(g(x^2))$.

$$j'(x) = f'(g(x^2))g'(x^2)2x$$

$$j'(2) = f'(2)g'(4)4 = 8$$

2. (14 points) The following is a graph of the **derivative** of f.
The function f is defined for all real numbers.



(a) For which values of x, if any, does f have a local maximum?

$x = -6, 2$

(b) For which values of x, if any, does f have a local minimum?

$x = -2$

(c) Which values of x, if any, are inflection points of f?

$x = -4, 0, 4, 6$

(d) Over which intervals is f increasing?

$(-\infty, -6), (-2, 2)$

(e) Over which intervals is f concave down?

$(-\infty, -4), (0, 4), (6, \infty)$

3. (2+8 points) The logistic model for population growth is a model that accounts for the fact that population cannot grow indefinitely. The formula for the logistic model is given by $P(t) = \dfrac{L}{1 + Ae^{-kt}}$ where L and A are positive constants.

(a) The carrying capacity is the horizontal asymptote of $P(t)$. What is the carrying capacity? What does this mean in practical terms?

The carrying capacity $= \lim_{t \to \infty} P(t) = L$. This gives an upper bound on the population, i.e., the population can approach a value of L but can never quite reach it.

(b) List the steps you would take to find the value of t for which the population is growing the fastest? Give reasons for each step. You do **NOT** have to carry out any of these steps!!!!

1. Find P' as this gives the rate of growth and is the function we are interested in maximizing.

2. Find P'' in order to find the critical points of P'.

3. Find the critical points of P' by determining where P'' is equal to zero or undefined.

4. Test the critical points of P' by either
   - taking the third derivative and testing the critical points,
   - looking at the sign of P'' around the critical points, or
   - giving a graphical argument based on the graph of P or P'.

4. (6 points) The shape of a balloon used by a clown for making a balloon animal can be approximated by a cylinder. As the balloon is inflated, assume that the radius is increasing by 2 cm/sec and the height is given by $h = 2r$. At what rate is air being blown into the balloon at the moment when the radius is 3 cm?

The formula for the volume of a cylinder with radius r and height h is given by $V = \pi r^2 h$. We know that $h = 2r$, so we can write $V = 2\pi r^3$. Taking the derivative with respect to t of both sides we get

$$\frac{dV}{dt} = 2\pi 3r^2 \frac{dr}{dt}.$$

We are interested at the time when $r = 3$ and $\frac{dr}{dt} = 2$, so

$$\frac{dV}{dt} = 108\pi \text{cm}^3/\text{sec}.$$

5. (8 points) In introductory physics one learns the formula $F = ma$, connecting the force on an object, F, with the mass of the object and the acceleration that the object experiences under the force. One also learns the formula $p = mv$ where p is the momentum of an object, m is the mass, and v is the velocity.

(a) Derive the formula $F = ma$ given that $\frac{dp}{dt} = F$, assuming that the mass is constant and that $p = mv$. Explain your answer.

Take the derivative of $p = mv$ with respect to t to get

$$\frac{dp}{dt} = m\frac{dv}{dt}$$

but since acceleration is the derivative of velocity, this gives

$$F = ma.$$

(b) Derive a formula for the force F if the mass is not assumed to be constant.

We do the same thing as in part (a), except this time mv is a product of two functions of t. Therefore we get

$$F = \frac{dp}{dt} = v\frac{dm}{dt} + ma.$$

6.(9+4 points) (a) Find the values of a and b so that the function $f(x) = axe^{-bx}$ has a local maximum at the point (3, 12).

$$f'(x) = abe^{-bx} - abxe^{-bx}$$

This is 0 when $a - abx = 0$. Therefore, either $a = 0$ or $x = \frac{1}{b}$. If $a = 0$, then $f(x) = 0$ for all x so cannot have a maximum at $(3, 12)$. So it must be that $x = \frac{1}{b}$. Since we want this maximum to occur at $x = 3$, $b = \frac{1}{3}$. Now using that the y-value at the maximum is 12, we have $12 = 3ae^{-1}$. Therefore, $a = 4e$.
Now to see this is actually a maximum we must take the second derivative:

$$f''(x) = -\frac{8}{3}e^{-\frac{1}{3}x+1} + \frac{4}{9}xe^{-\frac{1}{3}x+1}.$$

At $x = 3$, $f''(3) = -\frac{4}{3} < 0$. So we see this is a maximum of f.

(b) Does f have any inflection points for $x > 0$? If so, for what value(s) of x? If not, how do you know? [Use the function you found for part (a) here. Show your work or your reasoning.]

We have from part (a) that

$$f''(x) = \left(-\frac{8}{3} + \frac{4}{9}x\right)e^{-\frac{1}{3}x+1}.$$

This is equal to 0 if and only if $x = 6$. For $x < 6$ we have $f''(x) < 0$ and for $x > 6$ we have $f''(x) > 0$. Therefore $x = 6$ is an inflection point.

7. (4+3+3 points) **(a)** Determine the tangent line approximation for $f(x) = \sin x$ near $x = \frac{\pi}{3}$ (i.e., 60°).

$$f\left(\tfrac{\pi}{3}\right) = \sin\left(\tfrac{\pi}{3}\right) = \tfrac{1}{2}.$$
$$f'(x) = \cos x, \text{ so } f'\left(\tfrac{\pi}{3}\right) = \cos\left(\tfrac{\pi}{3} = \tfrac{\sqrt{3}}{2}\right).$$
$$\text{Thus, near } x = \tfrac{\pi}{3},\ f(x) \approx \left(\tfrac{\sqrt{3}}{2}\right) + \tfrac{1}{2}\left(x - \tfrac{\pi}{3}\right).$$

(b) Use your answer from part (a) to give an approximation of $\sin\left(\frac{31}{90}\pi\right)$ without using your calculator. Note that $\frac{31}{90}\pi = 62°$. Show your work.

$$\sin\left(\frac{31}{90}\pi\right) \approx \left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\left(\frac{31\pi}{90} - \frac{\pi}{3}\right) = \left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\left(\frac{\pi}{90}\right).$$

(c) Should your answer for part (b) be an over estimate or an under estimate? Justify your answer without indicating what $\sin\left(\frac{31}{90}\pi\right)$ is from your calculator.

The graph of $y = \sin x$ is concave down at $x = \frac{\pi}{3}$, so it is an overestimate as can easily been seen by sketching a graph of $\sin x$. [Or, could take the second derivative here and use the fact that $f''(\frac{\pi}{3}) < 0$.]

8. (2+10 points) Over the summer you are hired by a trucking company to help them improve operations. A truck driver is paid \$12 per hour for driving a truck over a 200 mile stretch of highway. The cost of driving the truck at an average velocity of v miles per hour is $(5 + .568v)$ dollars. The truck driver must drive between 40 mph and 70 mph.

(a) If the truck driver drives an average of v miles per hour for 200 miles, how long does he drive?

$$t = \frac{200}{v}\text{hours}$$

(b) At what average speed should the truck driver be told to drive in order to minimize the company's cost for the 200-mile trip? Note that the company's cost is the cost of paying the driver plus the cost of driving the truck.

The cost function is given by

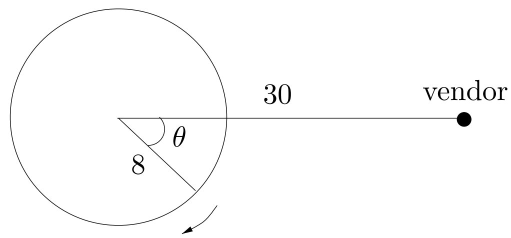
$$C(v) = 12\left(\frac{200}{v}\right) + (5 + 0.568v).$$

Since we would like to minimize this function, we take the derivative and set it equal to 0:

$$\begin{aligned} C'(v) &= -\frac{2400}{v^2} + .568 = 0 \\ v &= \pm 65\text{mph-(but we can discard the negative.)} \end{aligned}$$

To see that this is a minimum, observe that $C''(65) > 0$. Now we just need to check this against the endpoints. $C(65) = \$78.84$, $C(40) = \$87.72$, and $C(70) = \$79.04$. So the driver should be told to drive an average of 65 miles per hour in order to minimize the cost to the company.

9. (2+4+6 points) You have been searching for the cotton candy vendor all day at the carnival. As you board the merry-go-round, you spot the candy man. Unfortunately, you are stuck on the merry-go-round. The vendor's stand is 30 feet from the center of the merry-go-round, and you begin your ride directly on the line of sight between the center of the merry-go-round and the vendor. The merry-go-round has a radius of 8 feet and is turning at a rate of $\frac{\pi}{60}$ radians/second.



(a) How long does it take for the merry-go-round to rotate $\frac{\pi}{6}$ radians?

$t = 10$ seconds.

(b) How far are you from the vendor when the merry-go-round has rotated $\frac{\pi}{6}$ radians? [The law of cosines may help here. It states that given a triangle of side lengths a, b, and c with angle θ between sides a and b, then one has $c^2 = a^2 + b^2 - 2ab\cos\theta$.]

Use the law of cosines with $a = 8$, $b = 30$, $\theta = \frac{\pi}{6}$, and c the distance between you and the vendor. So $c = 23.42$ feet.

(c) How fast is the distance between you and the vendor changing when the merry-go-round has rotated $\frac{\pi}{6}$ radians?

Take the derivative of the law of cosines with respect to t:

$$2c\frac{dc}{dt} = 2ab\sin(\theta)\frac{d\theta}{dt}.$$

Solving this equation for $\frac{dc}{dt}$ and using that $a = 8$, $b = 30$, $c = 23.42$, $\theta = \frac{\pi}{6}$, and $\frac{d\theta}{dt} = \frac{\pi}{60}$ we obtain that $\frac{dc}{dt} = 0.27$ feet/second.