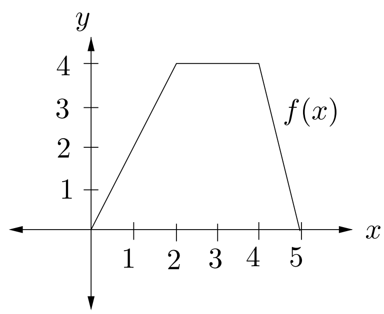
MATH 115 — FINAL EXAM SOLUTIONS

1. (12 points) Let $g(x)$ be a continuous function such that $\int_2^3 g(x)dx = 5$. Let $f(x)$ be given by the following graph:



(a) Find $f'(1)$.

$f'(1) = 2$.

(b) Find $\int_1^2 g(x+1)dx$.

$\int_1^2 g(x+1)dx = 5$.

(c) Find the average value of f on the interval $[0, 4]$.

The average value of f over $[0, 4] = 3$.

(d) Find $\int_2^3 (f(x) + 3g(x))dx$.

$\int_2^3 (f(x) + 3g(x))dx = 19$.

(e) If $G'(x) = g(x)$ and $G(2) = 7$, find $G(3)$.

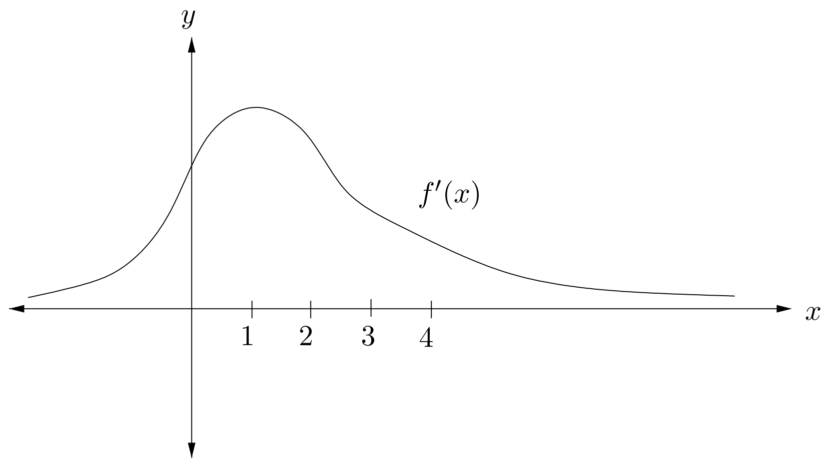
$G(3) = 12$.

(f) If $F'(x) = f(x)$, describe two graphical features of F on the interval $0 < x < 1$.

F is increasing and concave up.

2. (6 points) Using the graph of $f'(x)$ provided, list the following in increasing order:

$$\int_1^3 f'(x)dx, \qquad f(3) - f(2), \qquad f(2) - f(1)$$



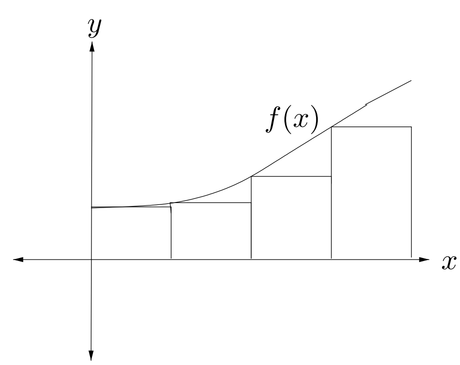
$\underline{f(3) - f(2)}$ less than $\underline{f(2) - f(1)}$ less than $\underline{\int_1^3 f'(x)\,dx}$

3. (5 points) **(a)** Briefly explain the difference between the indefinite integral $\int f(x)\,dx$ and the definite integral $\int_a^b f(x)\,dx$.

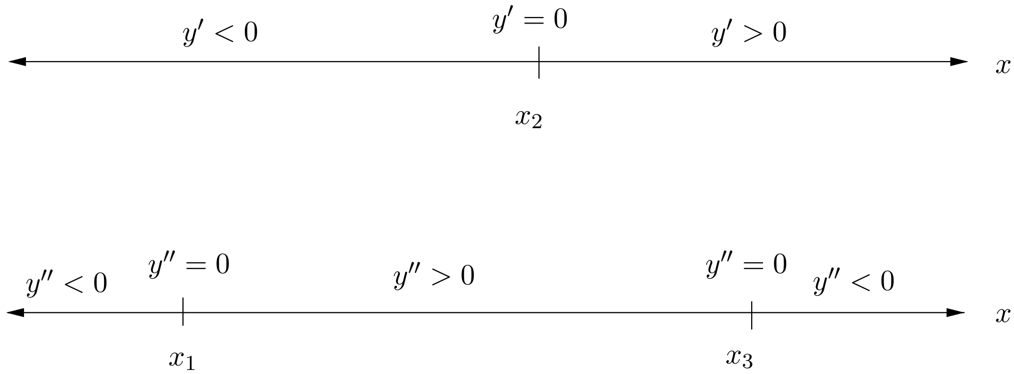
The indefinite integral is a function and the definite integral is a number.

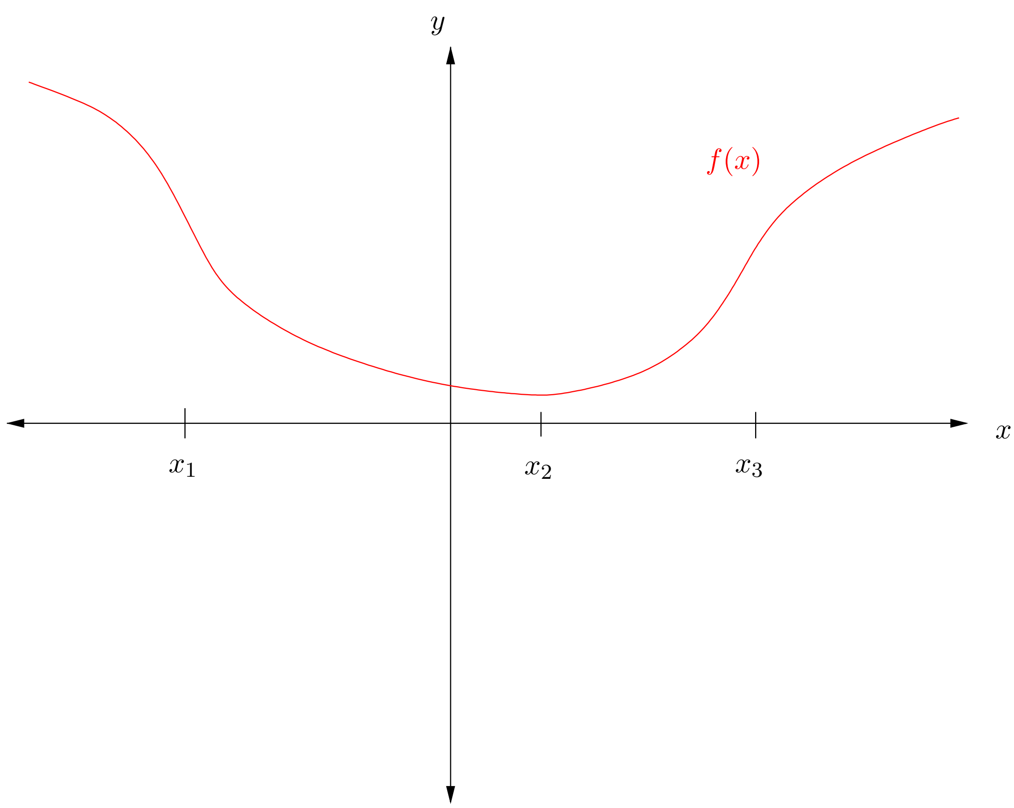
(b) What is the connection between a Riemann sum and one or more of the integrals in part (a)? Your answer should include a picture and a clear explanation.

A Riemann sum is an approximation to the definite integral. One forms rectangles as pictured and adds up the areas to approximate the area under the curve. As the width of the rectangles goes to zero the Riemann sum becomes a better and better approximation to the actual area under the curve.



4. (8 points) Sketch a possible graph of $y = f(x)$, using the given information about the derivatives $y' = f'(x)$ and $y'' = f''(x)$. Assume that f is defined and continuous for all real x. Label all local extrema and inflection points.





NOTE: there should be a local min labeled at x_2 and inflections points at x_1 and x_3

5. (4+6+3 points) Your uncle Harry absolutely LOVES eggnog around the holidays. The rate at which he drinks it at your family holiday party is given by the function $r(t)$ where t is measured in hours and $r(t)$ is in liters/hour. Suppose $t = 0$ corresponds to 6 pm when the party begins.

(a) Write a definite integral that represents the total amount of eggnog uncle Harry consumes between 8 pm and 2 am the next morning.

$$\int_2^8 r(t)\,dt$$

(b) If Uncle Harry's rate of eggnog drinking is given by $r(t) = e^{-t} + 1$, use a left hand sum with three (3) subdivisions to estimate the amount of nog Harry drinks in the first four hours of the party. Show all of your work.

$$r(0)\frac{4}{3} + r\left(\frac{4}{3}\right)\frac{4}{3} + r\left(\frac{8}{3}\right)\frac{4}{3} = 2\cdot\frac{4}{3} + (e^{-\frac{4}{3}} + 1)\frac{4}{3} + (e^{-\frac{8}{3}} + 1)\frac{4}{3}.$$

(c) Should your estimate in part (b) be an underestimate or an overestimate? Explain.

It is an overestimate because the function $r(t)$ is a decreasing function.

6. (9 points) You feel that you need to escape the cold climate before you return to school, so you decide to drain your savings account and head for somewhere warm and tropical. However, the cost of airline tickets seems to vary from day to day, moment to moment. At the eggnog party you meet a person who works in airline ticket pricing. He actually gives you a web site where you can check out the rate at which the cost of tickets on your preferred airline is changing at any given moment on a given day. (At this site, the rate of change is defined for all times, t.)

(a) If the rate of change is negative, would you be inclined to buy or wait? Why?

If the rate of change is negative, at this moment the ticket price is decreasing so you should wait.

(b) If the rate of change is equal to zero, what would that mean? Explain.

If the rate of change is zero then the ticket price is not changing at this moment. It could mean that the price function is at a max or min, or that the price is constant over a period of time.

(c) If you could get the formula for the data generated on the web site, would that help you to decide what to do in the case of part (b)? If so, how? If not, why not?

It would definitely help you as you could find a global minimum and then purchase the ticket at that price.

7. (4+2+3+6 points) You got a flight out, finally, but in your haste to leave, you locked Frosty the Snowman, Jr. in the warm greenhouse. Suppose $r(t)$ is the rate in cm^3/min that Frosty's volume is changing as he is trapped in the greenhouse. The time the doors of the greenhouse were closed corresponds to $t = 0$.

(a) Explain the meaning of the quantity $\int_2^5 r(t)dt$ in the context of this problem.

This is the total amount of volume Frosty has changed between 2 minutes after the door was closed up to 5 minutes. Since it only makes sense that Frosty is melting, we could say it is the amount of snow in cm^3 that Frosty melted between 2 and 5 minutes after the door was closed.

(b) What do you expect the sign of $r(t)$ to be for the meaningful domain of this problem? Why?

The sign should be negative indicating that Frosty is melting. It would not make sense for him to gain volume when he is in a warm greenhouse.

(c) If $r(t) = 3t^2 - 432$, what is the domain that makes sense for this problem? Why?

As explained in part (b) we want this to be negative. Therefore, we want $3t^2 - 432 < 0$. Solving this for t and noting that we would want a positive time, we need $0 \le t \le 12$ measured in minutes.

(d) Use the Fundamental Theorem of Calculus (and common sense) to determine the volume (in cm^3) of Frosty, Jr. when the door to the greenhouse was closed. Show all of your work and reasoning.

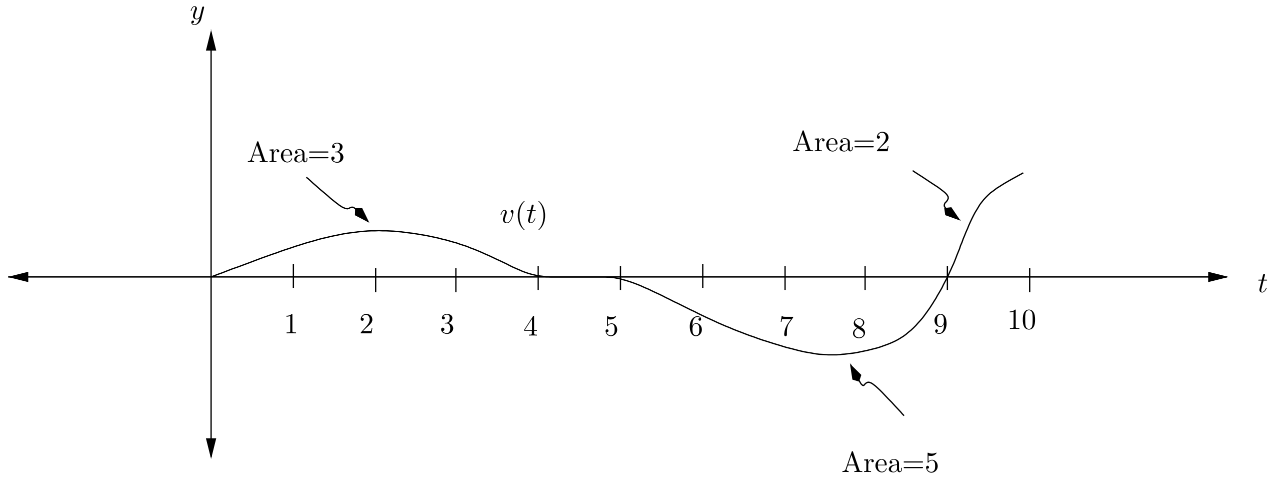
If we let $R(t)$ be the volume of Frosty, so that $R'(t) = r(t)$, then the Fundamental Theorem of Calculus gives us:

$$\int_0^{12} r(t)\,dt = R(12) - R(0).$$

We know that when $t = 12$ the domain ends, so we can assume that Frosty has completely melted at this point. (Don't worry, Santa can save him still!) Frosty's original volume is given by $R(0)$. So we have

$$\begin{aligned} R(0) &= -\int_0^{12} (3t^2 - 432)\,dt \\ &= -(t^3 - 432t)\,\big|_0^{12} \\ &= 3,456 \text{ cm}^3. \end{aligned}$$

8. (3+3+3 points) Finally in your tropical paradise, you are strolling through the rain forest when you come upon a hummingbird. He is flitting up and down a vine of flowers. The graph below gives the bird's **vertical** velocity (ft/sec) as a function of time (sec). Positive velocity indicates he is going up, and negative velocity indicates down.



(a) At which time(s) is the hummingbird likely hovering at a flower? Explain how you arrived at your answer.

The hummingbird is likely hovering at a flower when there is an extended period of time of zero vertical velocity. This happens between 4 and 5 seconds.

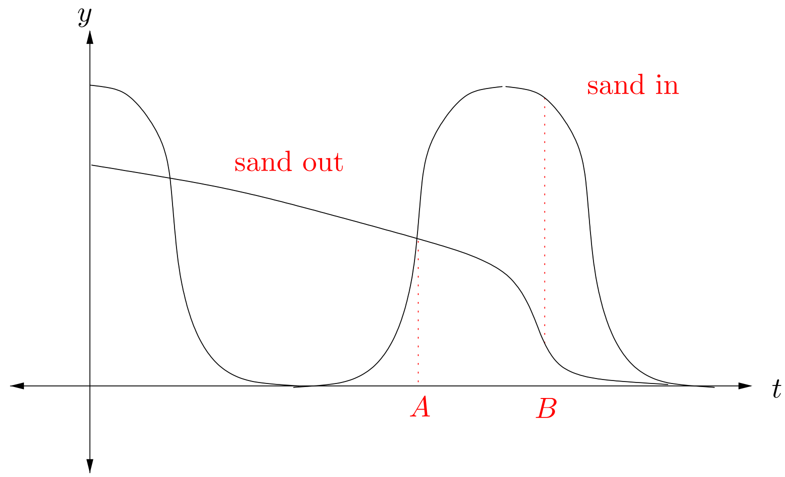
(b) At which time during the 10-second period is the hummingbird highest off the ground? Explain how you arrived at your answer.

The hummingbird's height off the ground is given by it's height at time zero plus the integral of $v(t)$ over the time. One sees the hummingbird is highest at time $t = 4$ seconds because one loses more height between 5 and 9 seconds then one gains between 9 and 10 seconds.

(c) At which time(s) is the hummingbird's vertical acceleration the greatest? Explain how you arrived at your answer.

The vertical acceleration is the derivative of the velocity function, so one wants to see where the largest derivative of $v(t)$ is. It should be clear that this occurs at $t = 9$ seconds.

9. (4+3+3 points) After watching hummingbirds for a while, you head to the beach to tan. Shortly after you begin tanning you are overcome with boredom. You decide to pass your time by digging a glorious hole in the ground. Just as a satisfactory hole begins to emerge from your efforts, the tide comes in and waves begin to wash sand back into your hole. You continue digging, but your enthusiasm is ending. Finally, you realize you're defeated and give up. The rate of sand flow into the hole due to the incoming waves and the rate of flow of sand out of the hole due to your digging are graphed below. Assume time $t = 0$ corresponds to the first wave coming in.



(a) Label which curve corresponds to the rate of sand going out of the hole and which curve corresponds to the rate of sand being washed into the hole by the waves. Explain how your arrived at your answer.

As the waves begin to wash in you tire and eventually give up. This should correspond to a rate out of the hole that is slowly decreasing over time and eventually drops to zero.
Waves wash sand into the hole at a high rate as they wash in, then the rate drops to zero as the wave recedes. This pattern repeats with each wave. This is why the graphs are labelled as they are.

(b) Mark and label the point on the graph when the least amount of sand was in the hole. Label the point as A. Explain how you arrived at your answer.

The points to consider here is $t = 0$ as the first wave washes in and the point marked A on the graph above. After point A more sand is going into the hole then you are shovelling out, so clearly the hole is filling at this point. One sees that the minimum amount of sand must be in the hole at point A because the volume between rate out and rate in between 0 and A is positive, so you have removed more sand over this period then the waves have washed in.

(c) Mark and label the point on the graph when the amount of sand in the hole was increasing most rapidly? Label the point as B. Explain how you arrived at your answer.

The amount of sand is increasing most rapidly when the difference between sand in - sand out is the largest.

10. (13 points) Alas, you returned home without a cent to your name–but you had a great time! You are hitching a ride back to school with friends. You have agreed to take a small suitcase in order to save space, but you realize that you also need to take all the souvenirs that you are bringing back to your friends at school. You determine that you need 3000 in^3 of space for those gifts. Your friend agrees to strap a box to the roof of his car. You are worried about the straps crushing the box and about the weather, so you decide to have a box made. The box-making place agrees to make a rectangular box with reinforced sides and weatherproofing on the top and sides. The sides come in a standard height of 6 inches and cost \$0.25 per in^2. The top costs \$0.15 per in^2, and the materials for the bottom cost \$0.05 per in^2. There is also a \$10 charge for labor.

(a) You are going to have to go into debt to get this box, so find the dimensions of the box with the standard height sides (6 inches) that will minimize the cost.

We begin with the first condition. If the box has dimensions $6 \times l \times w$, then the volume of the box is given by $V = 6lw = 3000$. Solving for w we have $w = \frac{500}{l}$. Next we need to find the cost formula. Using the prices given in the problem and what we have for w we get:

$$\begin{aligned} C &= .15lw + .25[12w + 12l] + 0.05lw \\ &= .15 \cdot 500 + .25\left[\frac{12 \cdot 500}{l} + 12l\right] + 0.05 \cdot 500 + 10. \end{aligned}$$

To minimize this we take the derivative with respect to l and set it equal to zero:

$$.25\left[-\frac{6000}{l^2} + 12\right] = 0.$$

Using that $l > 0$ we solve this to get $l = 22.36$ inches. Thus $w = \frac{500}{22.36} = 22.36$ inches. The physical situation easily shows us that the endpoints will not work as if $l = 0$ or $w = 0$ then we cannot satisfy the volume condition.

(b) How much is the box going to cost?

We merely need to plug $l = w = 22.36$ inches into the formula for cost:

$$C(22.36) = \$244.16.$$