MATH 115 — FIRST MIDTERM EXAM

October 11, 2005

NAME: SOLUTION KEY

INSTRUCTOR: ______________________ SECTION NO: __________

1. Do not open this exam until you are told to begin.
2. This exam has 9 pages including this cover. There are 10 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed 2 sides of a 3 by 5 note card.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones and other sound devices, and remove all headphones.

PROBLEM	POINTS	SCORE
1	12	
2	12	
3	4	
4	12	
5	12	
6	10	
7	12	
8	12	
9	5	
10	9	
TOTAL	100	

1. (2 points each, no partial credit) Circle "True" or "False" for each of the following problems. Circle "True" only if the statement is *always* true. No explanation is necessary.

(a) If A and B are positive constants, then the function $f(x) = \log(|Ax + B|)$ has a vertical asymptote at $x = -B/A$.

$\boxed{\text{True}}$ False

(b) If an exponential function of t, in years, has decreased to 60% of the original value in two years, in four years it will decrease to 30% of the original value.

True $\boxed{\text{False}}$

(c) If $h(x) = 1.3(0.5)^x$ then the derivative, h', is decreasing for all x.

True $\boxed{\text{False}}$

(d) The functions $\sin(e^x)$ and $e^{sin(x)}$ are inverses of each other.

True $\boxed{\text{False}}$

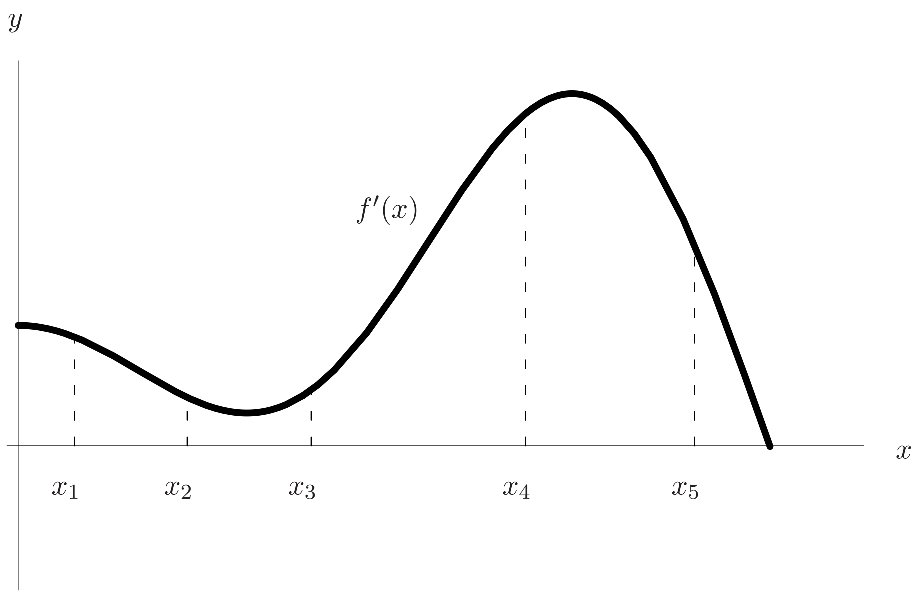
(e) If w is a continuous function for all x, then $\lim_{h\to 0} \dfrac{w(x+h) - w(x)}{h}$ exists for all x.

True $\boxed{\text{False}}$

(f) If $f''(x) > 0$ on the interval $[a, b]$, then the average rate of change of $f(x)$ on the interval $[a, b]$ is greater than $f'(x)$ for all $a < x < b$.

True $\boxed{\text{False}}$

2. (12 points) The graph of the *derivative* function, f', is given below. List all of the marked x-values, if any, from the figure for which the following statements are true. If no marked x-values apply, write "none."



(a) The value of $f(x)$ is greatest $\underline{\quad x_5 \quad}$

(b) $f''(x) < 0$ $\underline{\quad x_1, x_2, x_5 \quad}$

(c) f is decreasing ___none___

(d) Slope of f is positive $\underline{\quad x_1, x_2, x_3, x_4, x_5 \quad}$

(e) The graph of f is concave up $\underline{\quad x_3, x_4 \quad}$

3. (4 points) This exam will be graded out of 100 points. There are approximately 2000 students taking the exam. When the test has has been graded, there will be a function assigning to each student a score on the exam. Will this function be invertible? Why or why not?

No, this function will not be invertible. Since there are more students taking the exam than possible grades, there will be at least one grade assigned to two or more students. Since we cannot meaningfully assign one output (i.e., one student) to each grade, there can be no inverse function.

4. (12 points) The cost of gasoline has risen dramatically in the last six months. At the beginning of March, the cost of gasoline was \$2.10 per gallon, but at the beginning of September, the cost was \$3.00 per gallon.

(a) Suppose the cost of gasoline, C, measured in dollars per gallon is a linear function of time, t. Find a formula for cost of gasoline as a function t, in months, since the beginning of March.

Since $t = 0$ represents March, we have the vertical intercept of 2.10. The slope can be found by

$$\frac{\Delta C}{\Delta t} = \frac{3.00 - 2.10}{6} = \frac{.90}{6} = 0.15$$

Thus, $C(t) = 0.15t + 2.10$.

(b) Suppose further that you drive 200 miles per month and that your car averages 27 miles per gallon. Use your formula from part (a) to calculate the price of gasoline at the beginning of December. Assuming that the cost stays the same throughout the month of December, calculate your gas cost for the month of December.

In December, $t = 9$, so using our formula from (a), the cost of gas at the beginning of December will be $C(9) = \$3.45$ per gallon. We will use $\frac{200}{27} = 7.41$ gallons. Therefore, our total cost will be

$$7.41 \times \$3.45 = \$25.56$$

(c) Now suppose instead that C is an exponential function of time. Find a formula for the cost of gas as a function of t, in months since March.

The form of our function is:

$$C(t) = C_0 B^t$$

with $C_0 = 2.10$. Using the point $(6, 3.00)$, we have

$$3 = 2.10(B)^6$$

which gives $B = 1.0612$. Thus, if the increase is exponential, $C(t) = 2.10(1.0612)^t$.

(d) Use your formula from part (c) to calculate the cost of gasoline at the beginning of December. Assuming that this cost remains the same throughout December, use the milage information from part (b) to calculate your total gasoline cost for December.

Using the formula from (c) we find that the cost of gasoline at the beginning of December is $C(9) = \$3.58$ per gallon. Therefore, the total cost is

$$7.41 \times \$3.58 = \$26.56$$

5. (12 points) In Ann Arbor the earliest sunset is at 4 p.m. and the latest at 8 p.m. (ignoring daylight savings time).

(a) Determine a trigonometric function, f, as a function of t in days, where $f(t)$ gives the number of hours past midnight when sunset occurs. Assume that $t = 0$ represents the winter solstice (December 21) and ignore leap years. [Recall that winter solstice is the shortest day of each year.]

The period is clearly 365 days so that $B = \frac{2\pi}{365}$. The minimum and maximum values are 16 and 20, respectively so that $|A| = 2$ and $k = 18$ is the midline. Therefore, f can be given by:

$$f(t) = -2\cos(\frac{2\pi}{365}t) + 18$$

(b) Give a practical interpretation of $f(90)$ in the context of this problem.

The expression $f(90)$ represents the time in hours after midnight that the sun will sun will set 90 days after the winter solstice.

(c) Interpret $f'(120) = 0.03$ in the context of this problem.

The sun will set approximately 0.03 hours (1.8 minutes) later on the 121st day after the winter solstice than on the 120th day after the winter solstice.

(d) Suppose $g(x) = cf(x + h) - k$ for positive constants c, h and k. Give the following for $g(x)$ (your answers may involve c, h and k):

We can evaluate $cf(x + h) - k$ directly, using our formula from (a). We get:

$$g(x) = -2c\cos(\frac{2\pi}{365}(t + h)) + 18c - k$$

. Using this expression we can read off the answers to the following questions below.

(i) Amplitude 2c

(ii) Midline 18c-k

(iii) Period 365

6. (10 points)

(a) Suppose that $h(x) = g(f(x))$. Fill in the missing values based on the information given in the table:

x	1	2	3
$f(x)$	-1	1	3
$g(x)$	2/3	4/3	8/3
$h(x)$	1/6	2/3	8/3

(b) Which, if any, of f, g and h could be linear functions? Show evidence for your choice(s).

The function f could be linear, because the rates of change are constant:

$$\frac{3-1}{3-2} = \frac{1-(-1)}{2-1} = 2.$$

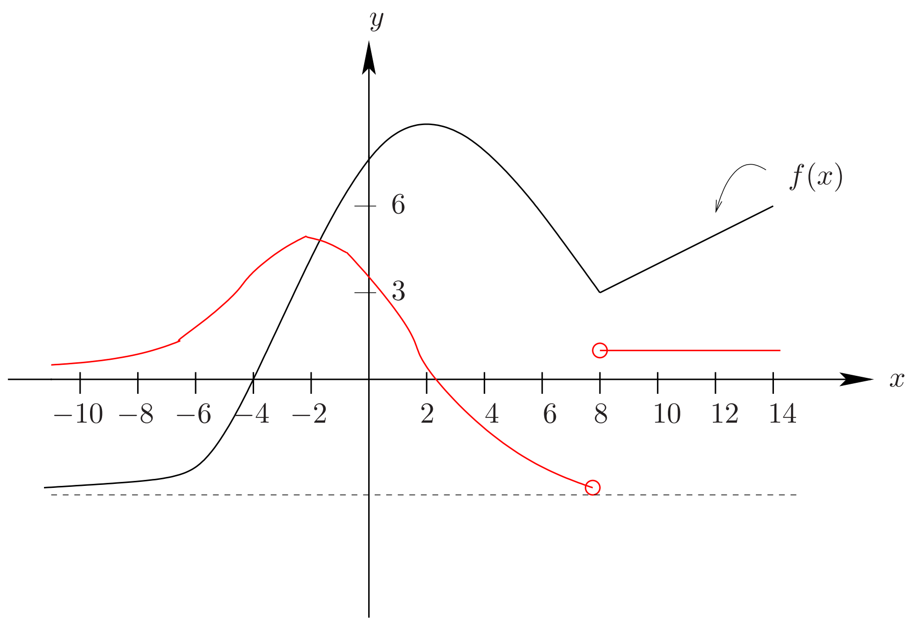
(c) Which, if any, of f, g and h could be exponential functions? Show evidence for your choice(s).

The functions g and h could be exponential, because the percent rates of change are constant. For Δx consistently one, we have

for $g(x) : \frac{8/3}{4/3} = \frac{4/3}{2/3} = 2$

and for $h(x) : \frac{8/3}{2/3} = \frac{2/3}{1/6} = 4.$

7. (12 points) The graph of a function f is given below.



(a) On the same set of axes, draw a graph of the derivative, $f'(x)$.

(b) Determine $f''(12)$.

$f''(12) = 0$

(c) Describe in words what the expression $\dfrac{f(-2) - f(4)}{-6}$ represents graphically.

The given quotient represents the slope of the line connecting the points $(-2, f(-2))$ and $(4, f(4))$.

(d) Write the following slopes in increasing order:

$$\frac{f(2)}{2} \qquad \frac{f(14) - f(8)}{14 - 8} \qquad \frac{f(4)}{4}$$

We may interpret each as the slope of a line. We get

$$\frac{f(14) - f(8)}{14 - 8} < \frac{f(4)}{4} < \frac{f(2)}{2}.$$

8. (12 points) The potential energy E, in joules, of an object above the Earth's surface is a function of the distance, h, in meters, of the object from the surface of the Earth. That is, $E = f(h)$.

(a) In the context of this problem, explain the meaning of $f(20) = 1000$?

The potential energy of an object 20 meters above the Earth's surface is 1000 Joules.

(b) In the context of this problem, explain the meaning of $f'(9) = 50$?

When an object is 9 meters above the Earth's surface, increasing the height of the object one meter, the potential energy will increase by approximately 50 Joules.

(c) In the context of this problem, explain the meaning of $f^{-1}(150) = 3$?

An object with 150 Joules of potential energy is 3 meters above the surface of the Earth.

(d) In the context of this problem, explain the meaning of $(f^{-1})'(400) = \frac{1}{50}$?

When an object has 400 Joules of potential energy, its height above the surface of the Earth must be increased by approximately $\frac{1}{50}$ of a meter to increase the potential energy by 1 Joule.

9. (5 points) Write the **limit definition** for the derivative of $e^{\sin(x)}$ with respect to x. (No need to simplify or to attempt to find the limit.)

$$\lim_{h\to 0} \frac{e^{\sin(x+h)} - e^{\sin(x)}}{h}$$

10. (9 points) Suppose

$$f(x) = \begin{cases} e^{\sin(x)} & x < \frac{\pi}{2} \\ kx & x \geq \frac{\pi}{2} \end{cases}$$

where k is some constant.

(a) If f is continuous, what is the value of k?

To ensure continuity, the two branches must have the same value at $x = \frac{\pi}{2}$. Therefore, $e^{\sin(\pi/2)} = k\frac{\pi}{2}$ is forced. Solving gives $k = \frac{2e}{\pi}$.

(b) Compute the average rate of change of f between $x = 1.5$ and $x = \frac{\pi}{2}$.

The number we want is

$$\frac{\Delta f}{\Delta x} = \frac{e - e^{\sin(1.5)}}{\pi/2 - 1.5} = .09606$$

(c) Compute the average rate of change of f between $x = 1.57$ and $x = \frac{\pi}{2}$.

The number we want is

$$\frac{\Delta f}{\Delta x} = \frac{e - e^{\sin(1.57)}}{\pi/2 - 1.57} = .00108$$

(d) Do you think f is differentiable at $x = \frac{\pi}{2}$? Explain your answer. [Your work from parts (a) - (c) may be useful here.]

The function will not be differentiable. The rate of change of f at $\frac{\pi}{2}$ approaches 0 from the left, but from the right it is $k = \frac{2e}{\pi} \neq 0$. Therefore, the function has a cusp or sharp corner at $x = \frac{\pi}{2}$.