MATH 115 — SECOND MIDTERM EXAM

November 22, 2005

NAME: SOLUTION KEY

INSTRUCTOR: ______________________ SECTION NO: __________

1. Do not open this exam until you are told to begin.
2. This exam has 10 pages including this cover. There are 10 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed 2 sides of a 3 by 5 note card.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones and other sound devices, and remove all headphones.

PROBLEM	POINTS	SCORE
1	16	
2	9	
3	12	
4	10	
5	10	
6	6	
7	6	
8	10	
9	9	
10	12	
TOTAL	100	

1. (16 points) Use the information given below to answer the following questions. Show work where appropriate.

x	0	1	2	3	4
$f(x)$	0.5	2	2.5	0	-4
$f'(x)$	1.5	0.5	-1	-3	-3.5

(a) If $g(x) = Ax^2$ for some constant A, find $h'(2)$ where $h(x) = g(x) + f(2x)$. Your answer may involve the constant A.

We have $h'(x) = g'(x) + 2f'(2x)$ and therefore, $h'(2) = g'(2) + 2f'(4)$. Also, $g'(2) = 4A$, so $h'(2) = 4A - 7$.

(b) Suppose $k(x) = 4^{f(x)}$. Find $k'(1)$.

By the chain rule, $k'(x) = ln(4) \cdot f'(x) \cdot 4^{f(x)}$ and thus, $k'(1) = ln(4) \cdot f'(1) \cdot 4^{f(1)} = 16 \cdot 0.5 \cdot ln(4) = 8ln(4)$.

(c) Suppose $l(x)$ is a linear function of x, $l(4) = 0$, and $l'(4) < f'(4)$. Which of the following is true about $l(x)$? (Circle all that apply; you need not justify your answer):

(i) $l(x) > 0$ for $x > 4$.

(ii) $l(x) < 0$ for $x > 4$.

(iii) $l(x)$ is increasing for all x.

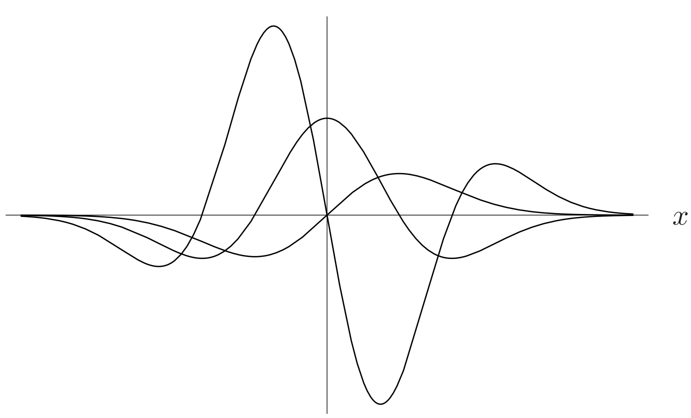
(iv) $l(x)$ is decreasing for all x.

Since $l'(4) < f'(4) = -3.5$, we see that $l(x)$ is decreasing. Since $l(x)$ is a line it is decreasing for all x. Because $l(4) = 0$, it also follows that $l(x) < 0$ for $x > 4$, so (ii) and (iv) are correct.

(d) Suppose $j(x)$ is an exponential function and that $j(0) = 1$. Let $h(x) = j(x)f(x)$. If $h'(0) = 7$, find a formula for $j(x)$.

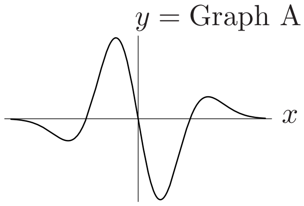
Because $j(x)$ is an exponential with $j(0) = 1$ we know $j(x) = b^x$ for some constant b. By the product rule, $h'(x) = j'(x)f(x) + j(x)f'(x)$, so we have an equation $7 = j'(0) \cdot 0.5 + 1.5$. It follows that $j'(0) = 11$. But $j'(0) = \ln(b)b^0 = \ln(b)$. So $b = e^{11}$ and therefore $j(x) = e^{11x}$.

2. (9 points) In the figure below, are the graphs of three functions, f, f', and f''. In the smaller figures, each graph is shown alone. To the right of the lower graphs give a clear explanation of how you determined which graph is f, which is f', and which is f''.

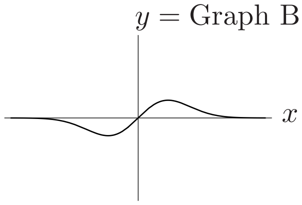


Explanation: Note that graph B cannot be the graph of the derivative of either other function, because B has only one zero. Thus, B must be f. Graph B has two local extrema, and Graph C has two zeros–and C is negative where B is decreasing and positive where B is increasing. Thus, C if f'. Graph C has three local extrema and those correspond to the zeros of Graph A. Plus, Graph A represents the increasing/decreasing behavior of C–and thus is f''.

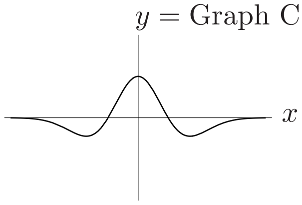
$y =$ Graph A



$y =$ Graph B



$y =$ Graph C



The graph of f corresponds to Graph ______B______

The graph of f' corresponds to Graph ______C______

The graph of f'' corresponds to Graph ______A______

3. (12 points) For the function $f(x) = ax^4 - 3x^3$ with constant $a > 0$, use the techniques of calculus to answer the following. Show your work and proper justification for your answers.

(a) Determine all critical points of f. Classify each as a local maximum, a local minimum, or neither.

First, $f'(x) = 4ax^3 - 9x^2 = x^2(4ax - 9)$ which is everywhere defined. So the critical points are the zeroes which are $x = 0$ and $x = \frac{9}{4a}$. Since the long term behavior of f' is like $4ax^3$, we see that $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > \frac{9}{4a}$.
For $0 < x < \frac{9}{4a}$, note that the sign of $f'(x)$ is determined by the sign of $4ax - 9$ because x^2 is always positive. Since $4ax - 9$ is just a line with positive slope (recall $a > 0$) and x-intercept $\frac{9}{4a}$ we must have that $4ax - 9 < 0$ for $x < \frac{9}{4a}$. It follows that $f'(x) < 0$ for $0 < x < \frac{9}{4a}$. Using the first derivative test, $x = \frac{9}{4a}$ is a local minimum and $x = 0$ is neither.

(b) Determine any global maxima or minima (if any).

The long run behavior of $f(x)$ is like ax^4 so we see that $f(x) \to \infty$ as $x \to \pm\infty$. Therefore $f(x)$ has no global maximum, but $x = \frac{9}{4a}$ is the global minimum.

(c) Determine all (if any) inflection points.

The second derivative, $f''(x) = 12ax^2 - 18x = x(12ax - 18)$. So the zeroes of the second derivative are $x = 0$ and $x = \frac{18}{12a} = \frac{3}{2a}$. Because the second derivative is a parabola opening upward, we conclude that $f''(x) < 0$ for $0 < x < \frac{3}{2a}$ and that $f''(x) > 0$ for $x < 0$ and $x > \frac{3}{2a}$. Thus both $x = 0$ and $x = \frac{18}{12a}$ are points of inflection.

4. (10 points) Every year pesticides used on adjacent agricultural land drain off into Lake Michigan. Eventually, scientists predict that the lake will become saturated with pesticides. As a result, the amount of pesticides in the lake $P(t)$ (in parts per million) is given as a function of time, t, in years since 2000, by

$$P(t) = a(1 - e^{-kt}) + b$$

where a, b and k are positive constants. Assume the saturation level of the lake for pesticides is 50 parts per million.

(a) If in the year 2000 the pesticide level of Lake Michigan was 5 parts per million, find a and b.

When k is a positive constant we have

$$\lim_{x\to\infty} e^{-kx} = \lim_{x\to\infty} \frac{1}{e^{kx}} = 0$$

and therefore,

$$\lim_{x\to\infty} P(x) = a(1-0) + b = a + b.$$

Since the saturation level is 50 parts per million, we get the equation $a + b = 50$. But also $P(0) = b = 5$ from the information given. It follows that $a = 45$.

(b) Find k if the pesticide level was increasing at a rate of 3 parts per million per year in the year 2000.

Since $P'(t) = 45ke^{-kt}$ we have that $P(0) = 45k$. So, $45k = 3$ hence, $k = \frac{1}{15}$.

(c) When the pesticide level reaches 30 parts per million, fish from the lake cannot be consumed by humans. In what year will the pesticide level in the lake reach 30 parts per million?

We must solve

$$P(t) = 45(1 - e^{-t/15}) + 5 = 30$$

for t. This equation becomes

$$e^{-t/15} = 1 - \frac{25}{45}$$

and therefore, $t \sim 12.16$ years. Thus, in the year 2012, the pesticide level should reach 30 ppm.

5. (10 points)

(a) If the graph of

$$\frac{a}{y} + x^2 + b \ln y = 6$$

goes through the point $(2, 1)$ for some implicitly defined y, find a.

Since $(2, 1)$ is on the curve it must satisfy the defining equation. Therefore,

$$a + 4 + 0 = 6$$

so $a = 2$.

(b) Suppose $g(x) = -4x + 9$ is the equation of the tangent to the curve defined above at the point $(2, 1)$. Find b.

The tangent line has a slope of -4 which tells us that $\frac{dy}{dx} = -4$ at the point $(2, 1)$. We can compute $\frac{dy}{dx}$ explicitly:

$$-ay^{-2}\frac{dy}{dx} + 2x + by^{-1}\frac{dy}{dx} = 0.$$

Plugging in $a = 2$ and solving for $\frac{dy}{dx}$ gives:

$$\frac{dy}{dx} = \frac{-2x}{by^{-1} - 2y^{-2}}.$$

So when we plug in $x = 2$, $y = 1$ and $\frac{dy}{dx} = -4$ we get the equation:

$$-4 = \frac{-4}{b - 2}$$

so that $b = 3$ follows.

6. (6 points) Consider the function $f(x) = 3xe^{ax} + x^2$, where a is a constant. If the error in the linear approximation to $f(x)$ near $x = 0$ is 0.02 when $x = 0.1$, what is a? Show your work.

First notice that $f(0) = 0$. We compute the derivative using the product and chain rules. We get:

$$f'(x) = 3e^{ax} + 3axe^{ax} + 2x$$

It follows that $f'(0) = 3$ and so the equation of the tangent line is $g(x) = 3x$. The error is defined as

$$Error = f(0.1) - \text{linear approximation at} x = 0.1$$

so plugging in $x = 0.1$, we get the following equation:

$$0.02 = (0.3e^{0.1a} + 0.01) - 0.3$$

When we solve this for a we find that $a \sim 0.3279$.

7. (6 points) The kinetic energy, K in Joules, of a particle in motion is a function of its fixed mass, M in kg, and its velocity, v, in $\frac{m}{s}$, and is given by:

$$K = \frac{1}{2}Mv^2.$$

For an object with a mass of 2 kg, how fast is its kinetic energy increasing when it is traveling $3\frac{m}{s}$ and accelerating at a rate of $10\frac{m}{s^2}$?

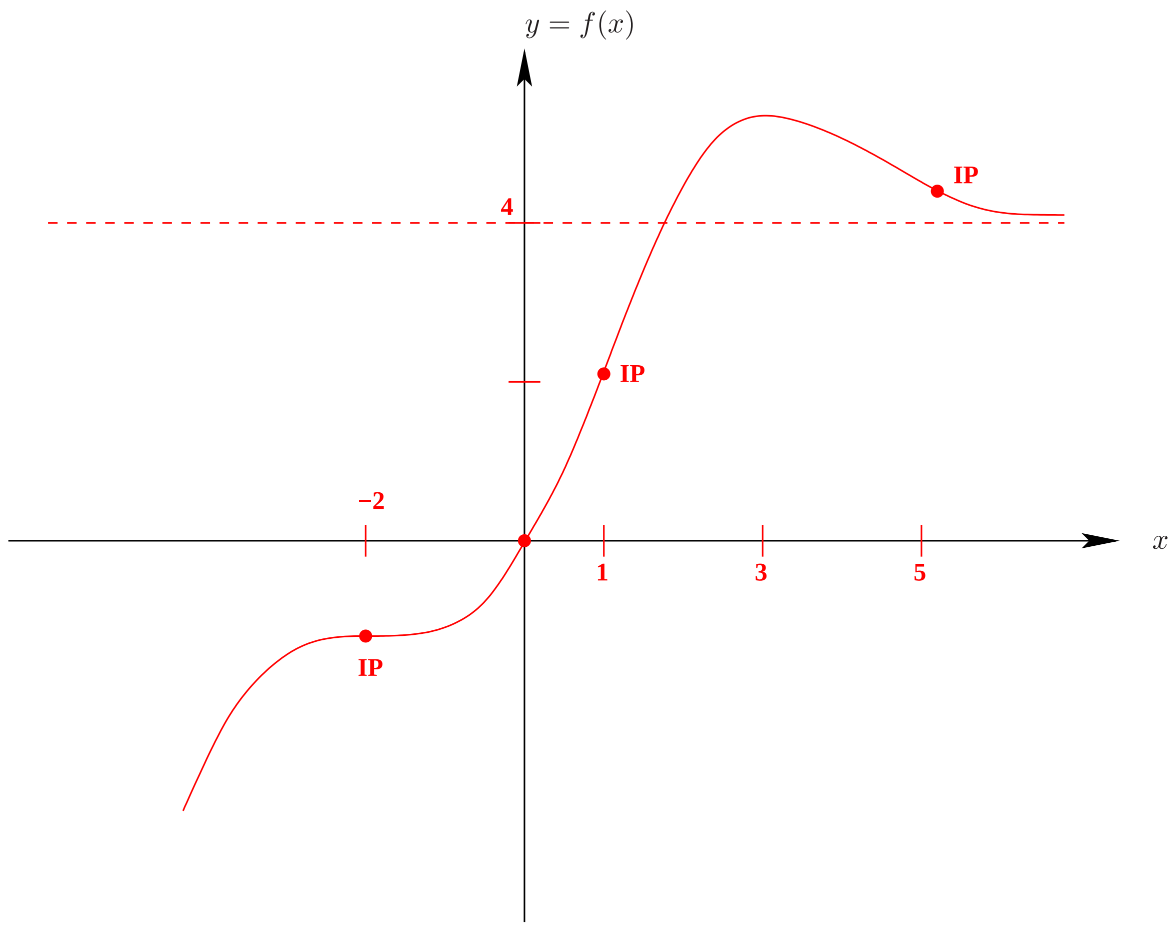
We differentiate the Kinetic energy equation with respect to time. Note that the mass, M, is fixed, and therefore is a constant with respect to time.

$$\frac{dK}{dt} = \frac{1}{2}2Mv\frac{dv}{dt} = Mv\frac{dv}{dt}$$

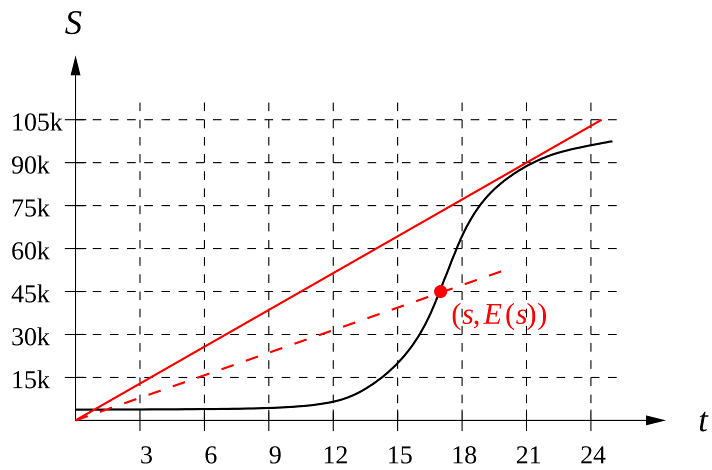
We now plug in $M = 2$, $v = 3$, $\frac{dv}{dt} = 10$ and solve for $\frac{dK}{dt}$. We get $\frac{dK}{dt} = 60$ Joules/sec (note that a Joule is the same as a $\frac{kg \cdot m^2}{s^2}$).

8. (10 points) On the axes below, sketch a graph of a single, continuous, twice differentiable function f with all of the following properties. Be sure to clearly label your axes.

- $f(0) = 0$ and $\lim_{x\to\infty} f(x) = 4$
- $f'(x) = 0$ for $x = -2, 3$
- $f'(x) \geq 0$ for $-\infty < x < 3$
- $f'(x) < 0$ for $x > 3$
- $f''(x) = 0$ for $x = -2, 1, 5$
- $f''(x) > 0$ for $-2 < x < 1$
- $f''(x) < 0$ for $-\infty < x < -2$ and $1 < x < 5$



9. (9 points) Suppose that the starting salary, $S(t)$, at U-M for someone in the lucrative mathematics exam-formatting and writing business is given by the following figure. The units of S are given as tens of thousands of dollars, and S is a function of t, the number of years of education attained by the exam-formatter and writer. Suppose that we consider value $V(t)$ of an education to be the starting salary per year spent in school.



(a) How is $V(t)$ related to $S(t)$?

Since $S(t)$ is the starting salary and t is the number of years spent in school, we can express $V(t)$ as $V(t) = \frac{S(t)}{t}$.

(b) How is $V(t)$ represented on the graph of $S(t)$ given?

It is the slope of the line connecting $(0, 0)$ and $(t, S(t))$–e.g. see the dotted line in the figure to a point $(s, E(s))$ on the curve.

(c) For $0 < t \leq 24$ what is the maximum of $V(t)$? According to the graph, how many years of education are the best value as far as starting salary is concerned? Explain.

Technically, there is no global max for $V(t)$ since $\frac{S(t)}{t} \to \infty$ as $t \to 0$. However, there is a local maximum where the line indicated in (b) has the largest slope, and this is where it is tangent to the curve–at approximately $t = 21$ years. Since we are all required to attend school to at least high school, this is probably more realistic as a "best value."

10. (12 points) You are in charge of ticket sales for the U-M/Ohio State football game next year. Fans can buy pre-season tickets prior to September 1, 2006 for \$22.50 each. After September 1st, the price will be \$25 per ticket. The \$25 tickets are called term tickets. It turns out that pre-season ticket sales are a good predictor of term ticket sales, though the relationship is somewhat complicated. The number of term tickets sold, $T(x)$ (in thousands), is a function of the number of pre-season tickets sold, x (in thousands), and is given by:

$$T(x) = -0.02x^2 + 1.9x + 8.$$

Assume that the maximum capacity of the stadium is 115,000. What number of pre-season and term tickets should be sold to maximize revenue? Be sure to completely justify your answers–using techniques of calculus–(i.e., merely a graph or table will not suffice).

Since x is the number of pre-season tickets sold and $T(x)$ is the number of term tickets sold, the total revenue, $R(x)$, is given by

$$R(x) = 22.5x + 25T(x).$$

We must find the global maximum of $R(x)$ on the interval $0 \leq x \leq 115$. We start by finding the critical points of R, and these occur where $R'(x) = 0$ (note that $R(x)$ is a quadratic polynomial so that it and it's derivative are defined everywhere). By direct calculation,

$$R'(x) = 22.5 + 25(-0.04x + 1.9) = 22.5 - x + 47.5$$

When we set $R'(x) = 0$ and solve for x we get $x = 70$, the single critical point of $R(x)$. Since $R(x)$ is a parabola opening *down*, $x = 70$ is in fact the global maximum of $R(x)$. Furthermore, $T(70) = 43$ (and observe that $70 + 43 = 113 < 115$ so that this falls within the stadium capacity). Therefore, to maximize revenue, U-M should sell $70,000$ pre-season tickets and $43,000$ term tickets.