MATH 115 — FINAL EXAM

December 15, 2005

NAME: Solution Key

INSTRUCTOR: ______________________ SECTION NO: __________

1. **Do not open this exam until you are told to begin.**
2. This exam has 9 pages including this cover. There are 9 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. **Include units in your answers where appropriate.**
6. You may use your calculator. You are also allowed two sides of one 3 by 5 note card.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.
8. Please turn **off** all cell phones and other sound devices, and remove all headphones.

PROBLEM	POINTS	SCORE
1	8	
2	9	
3	20	
4	8	
5	14	
6	10	
7	14	
8	10	
9	7	
TOTAL	100	

1. (2 points each) Circle "True" or "False" for each of the following problems. Circle "True" only if the statement is *always* true. No explanation is necessary.

(a) Suppose that a differentiable function h and its derivative, h', are continuous. If $h'(x) < 0$ for all $a \leq x \leq b$ then every left-hand sum estimate of $\int_a^b h(x)dx$ will be an overestimate.

$\boxed{\text{True}}$ False

(b) For $f(x)$ a continuous function, $\int_{-1}^{1} f(x)dx = 2\int_0^1 f(x)dx$.

True $\boxed{\text{False}}$

(c) If $\int_0^3 f(x)dx = 5$, then $\int_0^3 3f(x)dx = 15$.

$\boxed{\text{True}}$ False

(d) If $Z(t)$ is an anti-derivative for $z(t)$, then $Z(t+5)$ is also an anti-derivative for $z(t)$.

True $\boxed{\text{False}}$

2. (3 points each) Explain in words what the following represent:

(a) $\int_2^6 f(t)dt$ where $f(t)$ is the rate at which people are lining up outside of Target waiting for the store to open at 6 am, where t is in hours after midnight on the day after Thanksgiving,

$\int_2^6 f(t)dt$ is the total number of people who line up between 2:00 AM and 6:00AM.

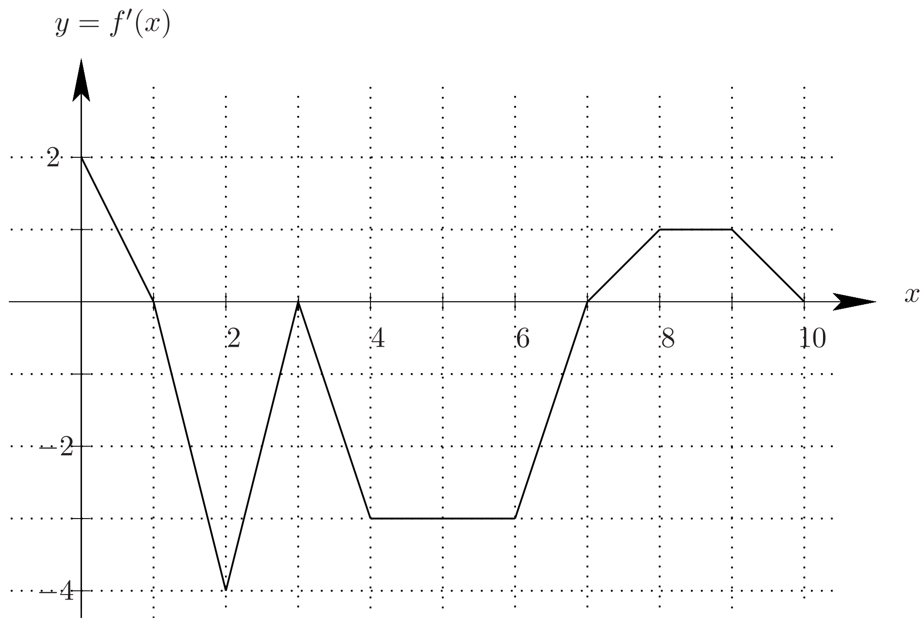
(b) $\int_0^4 a(t)dt$ where $a(t)$ is acceleration of an object in ft/sec^2 and t is in seconds

$\int_0^4 a(t)dt$ is the total change in velocity (in feet per second) of the object between the times $t = 0$ and $t = 4$.

(c) $\frac{1}{4}\int_5^9 r(t)dt$ where $r(t)$ is rainfall in inches per hour and t is in hours since noon

$\frac{1}{4}\int_5^9 r(t)dt$ is the average rainfall (in inches per hour) between 5:00 PM and 9:00 PM.

3. (20 points) Use the graph of $f'(x)$ on the closed interval $[0, 10]$ given in the figure below and the fact that $f(0) = 5$ to answer the following questions.



(a) What is the value of $f(3)$?

$f(3) = 2$

(b) For $0 \leq x \leq 10$, what x value(s) (if any) correspond to local maxima of f (if any)?

$x = 1, 10$

(c) For $0 \leq x \leq 10$, what x value(s) (if any) correspond to local minima of f (if any)?

$x = 0, 7$

(d) For $0 \leq x \leq 10$, what x value corresponds to the global minimum of f and what is the value of $f(x)$ at that point?

$x = 7$ $\qquad$ $f(x) = -7$

(e) If $H(x) = e^{f'(x)}$, find $H'(1.5)$.

$H'(1.5) = -4/e^2$

4. (8 points) Consider the function $f(x) = e^{-x^2}$.

(a) Use a right-hand sum with four equal subdivisions to estimate $\int_0^2 f(x)dx$.

With four subdivisions we must use $\Delta x = \frac{(2-0)}{4} = 0.5$. The right-hand sum estimate is therefore,

$$0.5(f(0.5) + f(1) + f(1.5) + f(2)) = 0.6352$$

(b) Without computing the integral from part (a), determine whether your estimate is an overestimate or an underestimate. Justify your answer.

Note that $f'(x) = -2xe^{-x^2}$. Since $e^{-x^2} > 0$ for all $x > 0$ and $-2x < 0$ for all $x > 0$, we see that $f'(x) < 0$ for all $x > 0$. Therefore, $f(x)$ is decreasing for all $x > 0$ and so the right-hand sum estimate is an underestimate.

5. (14 points) Suppose that the temperature (in degrees Celsius) on December 10th at the North Pole was described by the function $f(t) = -0.3t^2 + 7t - 38$, where t is hours after midnight for values of $0 \le t \le 24$.

(a) Find the average rate of change in temperature between 5 am and 2 pm. Show your work.

By definition, the average rate of change of $f(t)$ over the interval $5 \le t \le 14$ is

$$\frac{\Delta f}{\Delta t} = \frac{f(14) - f(5)}{14 - 5} = 1.3 \text{ degrees per hour}$$

(b) Find the average temperature between the hours of 5 am and 2 pm. Show your work.

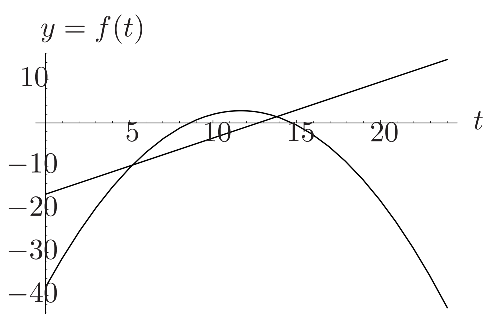
By definition, the average value of a function, f, on the interval $5 \le t \le 14$ is

$$\frac{1}{14-5} \int_5^{14} f(t)dt$$

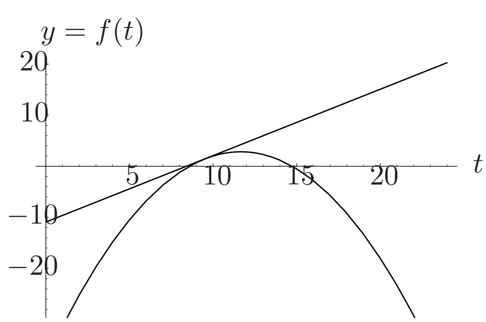
Note that $F(t) = -0.1t^3 + 3.5t^2 - 38t$ is an antiderivative for $f(t)$. Therefore, the average temperature is

$$\frac{F(14) - F(5)}{9} = -0.6 \text{ degrees}$$

(c and d) On the first sketch of $f(t)$ given in the figure below, show how the value from part (a) can be represented graphically. Use the second graph below, to approximate a time t for which $f'(t)$ is equal to the average rate of change of temperature from part (a). Show how this can be represented graphically. Carefully label and explain what you are indicating on each graph.



The value from part (a) can be represented as the **slope** of the line that joins the points $(5, -10)$ and $(14, 1.2)$. This line is shown in the figure above.



If f' is equal to the value from part (a), this can be represented graphically as the **slope** of a line tangent to the curve which has a slope of 1.3. This line is given in the graph above. The t value can be approximated on the graph.

6. (10 points) Using techniques from calculus, find the dimensions which will maximize the surface area of a solid circular cylinder whose height h and radius r, each in centimeters, are related by

$$h = 8 - \frac{r^2}{3}.$$

[Hint: the surface area of a cylinder is given by $2\pi r^2 + 2\pi r h$.]

Since the radius and height are related by $h = 8 - \frac{r^2}{3}$, the surface area may be written as a function of the radius, r, by

$$S(r) = 2\pi r^2 + 2\pi r(8 - \frac{r^2}{3}) = -\frac{2\pi}{3} r^3 + 2\pi r^2 + 16\pi r$$

To maximize the surface area we must find the global maximum of $S(r)$. We are only interested in positive values for r; i.e., the interval $r > 0$. We start by finding the critical points of $S(r)$. Note that $S'(r) = -2\pi r^2 + 4\pi r + 16\pi = -2\pi(r^2 - 2r - 8)$. To solve $S'(r) = 0$, we can factor. We get

$$-2\pi(r-4)(r+2) = 0$$

so the critical points occur at $r = 4$ and $r = -2$. On the interval $r > 0$, $r = 4$ is the only critical point. By the second derivative test, it is a local maximum ($S''(4) = -12\pi < 0$) Also, for very large values of r we see that $S(r)$ is negative so the global maximum occurs at $r = 4$. When $r = 4$ we find that $h = 8 - \frac{16}{3} = \frac{8}{3}$ and so these dimensions maximize the surface area.

$h = \frac{8}{3}$ cm

$r = 4$ cm

7. (14 points) Show your work!

(a) Confirm that

$$F(x) = \frac{1}{4}x^4 \ln(x) - \frac{1}{16}x^4 + 12$$

is an antiderivative for $f(x) = x^3 \ln(x)$, for values of $x > 0$. Show your work.

We must show that $F'(x) = f(x)$ for then $F(x)$ is an antiderivative for $f(x)$ by definition. To compute $F'(x)$ we use the product rule:

$$F'(x) = 4\frac{1}{4}x^3 \ln(x) + \frac{1}{4}x^4\frac{1}{x} - \frac{4}{16}x^3 = x^3 \ln(x) + \frac{1}{4}x^3 - \frac{1}{4}x^3 = x^3 \ln(x) = f(x).$$

(b) Use the Fundamental Theorem of Calculus to find $\int_1^2 x^3 \ln(x)dx$. Give your answer in *exact form*– i.e., not a decimal approximation.

Since $F(x)$ is an antiderivative for $f(x)$ by part (a), the Fundamental Theorem states that

$$\int_1^2 f(x)dx = F(2) - F(1).$$

By direct calculation,

$$F(2) = \frac{1}{4}16\ln(2) - \frac{1}{16}16 + 12 = 4\ln(2) + 11$$

while

$$F(1) = \frac{1}{4}\ln(1) - \frac{1}{16} + 12 = 12 - \frac{1}{16}.$$

Therefore,

$$\int_1^2 f(x)dx = 4\ln(2) - \frac{15}{16}$$

(c) Find an equation of the tangent to the graph of F at $x = 1$.

Since $F'(x) = f(x)$, the slope of the tangent line to $F(x)$ at $x = 1$ is $f(1) = 0$. We know from part (b) that $F(1) = 12 - \frac{1}{16}$, so the equation for the tangent line is

$$y = 12 - \frac{1}{16}$$

8.(10 points) On Christmas Eve, the Grinch and Santa each head **first** for Joe's house. The Grinch usually likes to arrive at houses after Santa, but for this first stop the Grinch wants to get to the cookies before Santa can. (The cookies at Joe's are *exceptionally* good.) Assume that Santa is directly North of the house (therefore traveling due South) while the Grinch is directly East of the house (traveling due West–also flying, so as to try to get ahead of Santa). Assume that both Santa and the Grinch are flying at the same altitude.

Santa is moving at 30 miles per hour, and the Grinch is going 28 miles per hour. How fast is the distance between them changing when Santa is 120 miles from Joe's house and the Grinch is 160 miles from the house?

We need to relate the distance, D, between Santa and the Grinch to the distance, x, between Santa and Joe's house and the distance, y, between the Grinch and Joe's house. The fact is that x, y and D form the legs of a right triangle, where D is the hypotenus, so

$$D = \sqrt{x^2 + y^2} = (x^2 + y^2)^{1/2}.$$

Now, x, y and D are all changing with time, t, so by the chain rule

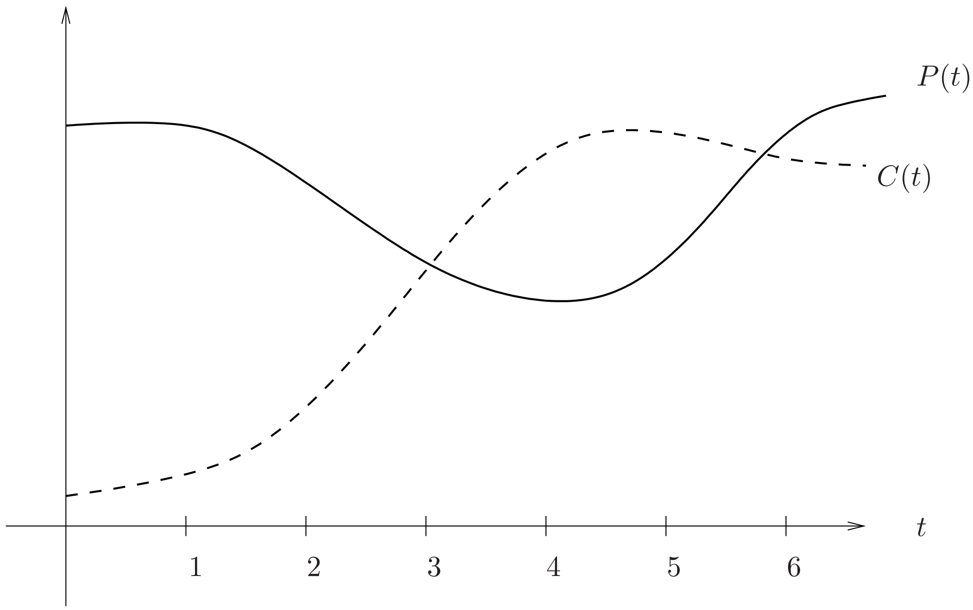
$$(*) \quad \frac{dD}{dt} = \frac{1}{2}(x^2 + y^2)^{-1/2}(2x\frac{dx}{dt} + 2y\frac{dy}{dt}).$$

Note that $\frac{dx}{dt} = -30$ (negative because Santa is moving towards Joe's house, so the distance between Santa and Joe's house is decreasing) and similarly, $\frac{dy}{dt} = -28$. When we plug $x = 120, y = 160, \frac{dx}{dt} = -30$, and $\frac{dy}{dt} = -28$ into $(*)$ we find that

$$\frac{dD}{dt} = -40.4$$

So the distance between Santa and the Grinch is decreasing at 40.4 miles per hour.

9.(7 points) In order to survive and perform their tasks, cells in your body must simultaneously produce and break down a molecule called ATP. When ATP is broken down, energy is released to the cell, and ATP is destroyed. For a certain cell, the rate of production of ATP, $P(t)$, in millions of molecules per second, and the rate at which ATP is broken down, $C(t)$, also in millions of molecules per second, are given in the following figure, where t is in seconds. The graph of $P(t)$ is shown as a solid line, and $C(t)$ is dashed.



Observe that since $P(t)$ is the rate at which ATP is being produced while $C(t)$ is the rate at which ATP is being broken down, $P(t)-C(t)$ is the rate at which ATP is accumulating in the cell. Therefore, the total change in ATP in the cell is represented graphically by the area between $P(t)$ and $C(t)$. Moreover, if $P(t) > C(t)$, then the area between them represents an increase in ATP while if $C(t) > P(t)$, the area between them represents a decrease in ATP. We will use this observation to answer the following.

(a) At time $t = 1$, is ATP increasing or decreasing?

Since $P(t) > C(t)$ at $t = 1$, we know that the rate that ATP is accumulating in the cell is positive at $t = 1$. Therefore the amount of ATP is *increasing* at $t = 1$.

(b) At approximately what time between $t = 0$ and $t = 6$ does the cell have the greatest amount of ATP? Explain.

Around $t = 3$. Note that $P(t) = C(t)$ around $t = 3$ and just before $t = 6$. Since $P(t) > C(t)$ for $0 \leq t \leq 3$, the area between them represents an increase in ATP. On the other hand, between $t = 3$ and $t = 6$, $C(t) > P(t)$ so the area between them represents a decrease in ATP. Therefore, the cell has the greatest amount of ATP at $t = 3$.

(c) At approximately what time between $t = 0$ and $t = 6$ is the amount of ATP in the cell decreasing the fastest? Explain.

Around $t = 4.5$ (between $t = 4$ and $t = 5$). We are looking for the time when $P(t) - C(t)$ is the most negative, (or where on the interval between $t = 3$ and $t = 6$ the slopes of tangents to C and P would be equal), and this occurs between $t = 4$ and $t = 5$.