MATH 115 – FIRST MIDTERM EXAM

October 10, 2006

NAME: SOLUTIONS

INSTRUCTOR: ____________ SECTION NUMBER: ____________

1. **Do not open this exam until you are told to begin.**
2. This exam has 10 pages including this cover. There are 9 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed two sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to show how you arrived at your solution.
8. Please turn off all cell phones and pagers and remove all headphones.

PROBLEM	POINTS	SCORE
1	9	
2	12	
3	12	
4	10	
5	8	
6	12	
7	11	
8	12	
9	14	
TOTAL	100	

1. (3 points each. No partial credit.) The questions on this page are multiple choice. They do not require an explanation. For each question, circle your choice for the correct answer(s). There may be more than one correct choice. Circle **ALL** answers that *must* be true about the given statement.

   (a) The function $h(x) = \dfrac{x^5 - x^3 + 2x^2}{x^3 - 3x^2}$

   (i) $\boxed{\text{is undefined for } x = 0 \text{ and } x = 3.}$

   (ii) has a horizontal asymptote of $y = -2/3$.

   (iii) has vertical asymptotes at $x = 0$ and $x = 3$.

   (iv) approaches $-\infty$ as x approaches ∞.

   (v) has a horizontal asymptote of $y = 1$.

   (b) If a function f is continuous at $x = a$, then

   (i) $\boxed{\lim_{x\to a} f(x) \text{ exists.}}$

   (ii) $\boxed{\lim_{x\to a} f(x) = f(a).}$

   (iii) $\lim_{x\to a} f(x) = f'(a)$.

   (iv) f is differentiable at $x = a$.

   (v) none of (i)-(iv) *must* be true.

   (c) If $\lim_{x\to 3} g(x) = 5$ for some function g, then

   (i) $g(3) = 5$.

   (ii) g is continuous at $x = 3$.

   (iii) $\boxed{\lim_{x\to 3^-} g(x) = \lim_{x\to 3^+} g(x).}$

   (iv) $g'(3) = 5$.

   (v) g is differentiable at $x = 3$.

2. Suppose $A(t)$ is a function that gives the average high temperature (in ^oF) in Ann Arbor as a function of t measured in months where $t = 0$ represents January (the coldest month in Ann Arbor).

   (a) (2 points) Puerto Montt, a city in Chile, is approximately the same distance from the equator as Ann Arbor, but it is in the southern hemisphere where the warmest month is January. Let $P(t)$ be a function that gives the average high temperature in ^oF in Puerto Montt as a function of time, t, in months. Write $P(t)$ in terms of $A(t)$.

   $P(t) = A(t-6)$ or, (equally acceptable) $P(t) = A(t+6)$

   (b) (2 points) The average high temperatures in Montreal are approximately 10^oF less than the average highs in Ann Arbor. If $M(t)$ is a function that gives the average high temperature in Montreal as a function of time, t in months, express $M(t)$ in terms of $A(t)$.

   $$M(t) = A(t) - 10$$

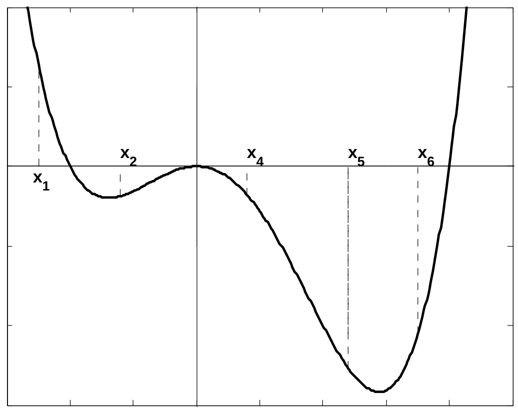
   (c) (5 points) The average high temperature in Ann Arbor ranges from a low of 30^oF in January to a high of 84^oF in July. Use this information to write $A(t)$ as trigonometric function.

   $$A(t) = -27\cos(\tfrac{\pi}{6}t) + 57$$

   (d) (1 point) What is the amplitude of the function found in (c)? 27

   (2 points) What is the period of the function found in (c)? 12 months

3. (12 points) Below is a graph of a function f:



(a) For which of the marked x values is $f'(x)$ the largest? x_6

(b) List all of the marked x values for which $f(x) > 0$. x_1

(c) List all of the marked x values for which $f'(x) < 0$. x_1, x_4, x_5

(d) List all of the marked x values for which $f'(x)$ is increasing. x_1, x_2, x_5, x_6

(e) List all of the marked x values for which $f(x)$ is increasing. x_2, x_6

(f) List all of the marked x values for which $f''(x)$ is negative. x_4

4. In 1950, a very invasive tree was introduced into a mountainous region of South Africa. Since then, the number of trees has grown exponentially. Suppose that $N(t)$ gives the number of trees, in *thousands*, as a function of time, t, measured in years since 1950.

   (a) (3 points) In the context of this problem, explain the meaning of $N^{-1}(25) = 46$.

   The equation $N^{-1}(25) = 46$ means that in 1996 there were 25,000 of the invasive trees in the region.

   (b) (3 points) In the context of this problem, explain the meaning of $N'(46) = 3$.

   In practical terms, $N'(46) = 3$ indicates that between 1996 and 1997, the number of invasive trees would increase by approximately 3000 trees.

   (c) (4 points) If 100 trees were introduced in 1950, use the above information to find a formula for $N(t)$.

   The equation will be of the form
   $$N(t) = N_o a^t$$
   where $N_o = 0.1$ and $N(t) = 25$ when $t = 46$.

   Thus,
   $$25 = 0.1a^{46},$$
   $$\text{so, } 250 = a^{46},$$
   which gives
   $$a = 250^{1/46} = 1.1275.$$

   An equation for $N(t)$ is then
   $$N(t) = 0.1(1.1275)^t.$$

5. (8 points) Below is a table of values for two functions f and g.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$f(x)$	-0.16	-0.284	-0.5	-0.64	0	0.64	0.5	0.28	0.16
$g(x)$	-0.88	-1.0888	-1	0.32	2	0.32	-1	-1.088	-0.88

Use the table to answer the following:

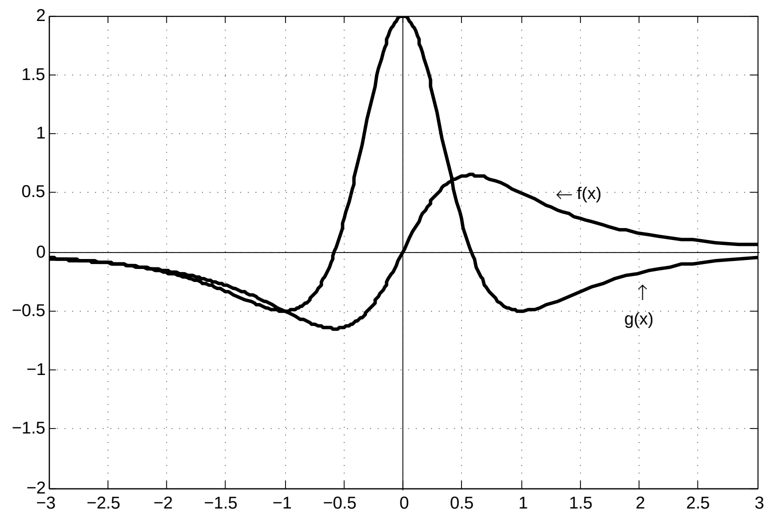
(a) $g(f(-1))$ $= g(-0.5) = 0.32$

(b) $3g(-1.5)$ $= 3(-1.0888) = -3.2664$

(c) $f(g(0))$ $= f(2) = 0.16$

(d) $g(2)f(-1)$ $= -0.88(-0.5) = 0.44$

Below is a plot of the functions f and g.



(e) Circle ***ONE*** of the following

$f(x) = g'(x)$ **or** $\boxed{g(x) = f'(x)}$,

AND explain your reasoning below.

The function f cannot be the derivative of g, because g is increasing for $x = -1$ to $x = 0$ (for example), and f is negative there. On the other hand, g is negative when f is decreasing, zero when f changes from decreasing to increasing (near $x = -0.5$), then positive when f is increasing, etc. Thus, g is the derivative of f.

6. (12 points) Consider the function $f(x) = \sin(x^2)$.

(a) Explain what the difference quotient $\dfrac{\sin(\sqrt{\pi}^2) - \sin(0)}{\sqrt{\pi}}$ represents.

The difference quotient represents the slope of the secant line joining the points $(0, 0)$ and $(\sqrt{\pi}^2, \sin(\sqrt{\pi}^2))$–or, it gives the average rate of change of the function $f(x) = \sin(x^2)$ between $x = 0$ and $x = \sqrt{\pi}$.

(b) Write the **limit definition** for $f'(\sqrt{\pi})$ without using the symbol f. No need to numerically evaluate the limit or approximate $f'(\sqrt{\pi})$.

The definition gives

$$f'(\sqrt{\pi}) = \lim_{h\to 0} \frac{\sin((\sqrt{\pi}+h)^2) - \sin(\sqrt{\pi}^2)}{h}.$$

(c) Suppose that g is a new function defined as follows:

$$g(x) = \begin{cases} 2f(x) & x < \sqrt{\pi/2} \\ kx + 4 & x \geq \sqrt{\pi/2} \end{cases} \qquad \text{for } f(x) \text{ as above}$$

For what value of k is the function g continuous?

Since both f and $kx + 4$ are continuous, we need to assure that the functions meet at $x = \sqrt{\pi/2}$. Thus, we need to solve for k if

$$2\sin(\sqrt{\pi/2}^2) = k\sqrt{\pi/2} + 4.$$

This gives

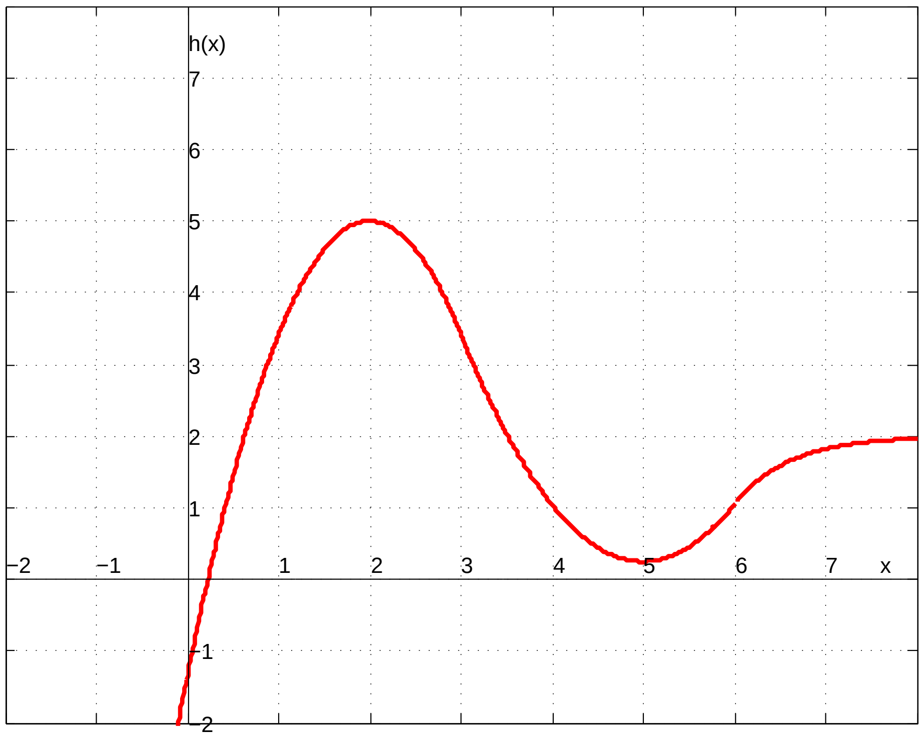
$$2 = k\sqrt{\pi/2} + 4 \qquad \text{(since } \sin(\sqrt{\pi/2}^2) = 1\text{)},$$

so

$$k = \frac{-2}{\sqrt{\pi/2}}.$$

7. (11 points) (a) On the axes below sketch a graph of a single, continuous differentiable function h that satisfies all of the following properties

- $h(2) = 5$
- $h''(x) < 0$ for $x < 3$
- $h'(5) = 0$
- $\lim_{x\to\infty} h(x) = 2$
- h' is positive for $x < 2$ and $x > 5$
- h is decreasing for $2 < x < 5$



(b) What is $\lim_{x\to-\infty} h(x)$? $-\infty$

(c) If $h'(0) = 2$, is it possible that $h'(-1) = 4$? Explain.

Yes, it is possible that $h'(-1) = 4$. We know is that h is increasing and concave down for $x < 0$, so we only need $h'(-1) > h'(0)$.

8. Astronauts travel to the moon and perform an experiment where they launch a special ball in the air. The ball is able to record its height above the surface of the moon at one second intervals, but when the ball lands it is damaged and the only information that the astronauts can recover is given in the following table:

time (seconds)	0	5	12	18	20	25	30	36	42	60
height (meters)	40	309.25	616.48	815.08	868	971.25	1033	1052.3	1011.9	532

Suppose $s(t) = h$ is the height of the ball as a function of time in seconds.

(a) (3 points) Compute the average velocity of the ball over the time interval $18 \leq t \leq 30$.

Average Velocity= $\dfrac{h(30) - h(18)}{30 - 18} = \dfrac{1033 - 815.08}{12} = 18.16$ m/s.

(b) (3 points) Approximate the instantaneous velocity of the ball when $t = 25$ seconds.

Any of the following are acceptable as an estimate:

From the left: $s'(25) \approx \dfrac{868 - 971.25}{-5} = 20.65$ m/s;

From the right: $s'(25) \approx \dfrac{1033 - 971.25}{5} = 12.35$ m/s;

Avg of Left and right: $s'(25) \approx \dfrac{1033 - 868}{10} = \dfrac{20.65 + 12.35}{2} = 16.5$ m/s.

(c) (3 points) Using the information in the table, approximate $s''(25)$. Be sure to carefully show your work. [Note: s'' is approximately constant for this function.]

We can use the information from part (b) to apprimate $s''(25)$:

$$s''(25) \approx \frac{s'(30) - s'(20)}{30 - 20} \approx \frac{12.35 - 20.65}{10} = -0.83 \ \ \text{m/s}^2.$$

(d) (3 points) If the rate of change from $t = 42$ to $t = 60$ were to remain constant, the ball would reach the surface of the moon at approximately $t = 80$. Use the information from the previous parts to decide if this is an overestimate or an underestimate of the time it takes for the ball to reach the ground. **Explain your answer.**

From part (c), s'' is negative (and we are told that s'' is basically constant over all intervals). Thus, the function is concave down. If we extrapolate from $t = 60$ to the vertical axis, we are assuming that the velocity is constant. Since the function s is concave down, the ball would actually hit the surface of the moon before $t = 80$. Thus, $t = 80$ is an overestimate.

9. Cliff is a stage manager for the Royal Shakespeare Company. He notices that the more people that are in the audience, the more nervous the actors get, and consequently they say their lines very fast and the play is shorter. Suppose $T(n)$ is a function that gives the running time of the play in minutes as a function of the number of people, n, in the audience. Cliff also remembers that the play ran 240 minutes at the last dress rehearsal when no one was in the audience and today they had a crowd of 300 people and the show ran for 2 hours.

   (a) (3 points) In the context of this problem what is the practical interpretation of $T(158)$?

   The expression $T(158)$ gives the number of minutes that the play will run when 158 people are in the audience.

   (b) (3 points) In the context of this problem what is the practical interpretation of $T'(158)$?

   The expression $T'(158)$ gives the approximate change in the number of minutes that the play will run when the audience increases from 158 to 159 people. The units of T' are minutes per person.

   (c) (4 points) Assuming that running time decreases linearly as the number of people attending increases, write an equation for the running time T in minutes as a function of n number of people attending.

   We are given two points, (0,240) and (300, 120). Thus, the slope of the linear function is

$$m = \frac{120 - 240}{300 - 0} = \frac{-120}{300} = -0.4.$$

   We are given the vertical intercept of (0,240), so an equation of the linear function is

$$T(n) = -0.4n + 240.$$

   (d) (4 points) Now assume instead that the running time is exponentially decreasing as a function of the number of the people in the audience, write an equation for the running time T in minutes as a function of the number of people n.

   In an equation of the form $T(n) = T_o a^n$, we are given that $T_o = 240$. Using the other given point, we have

$$120 = 240a^{300}$$

   so

$$a = 0.5^{1/300} = 0.9977$$

   Thus, if the running time is decreasing exponentially, an equation is

$$T(n) = 240(0.9977)^n.$$