Math 115 – Second Midterm Exam

November 14, 2006

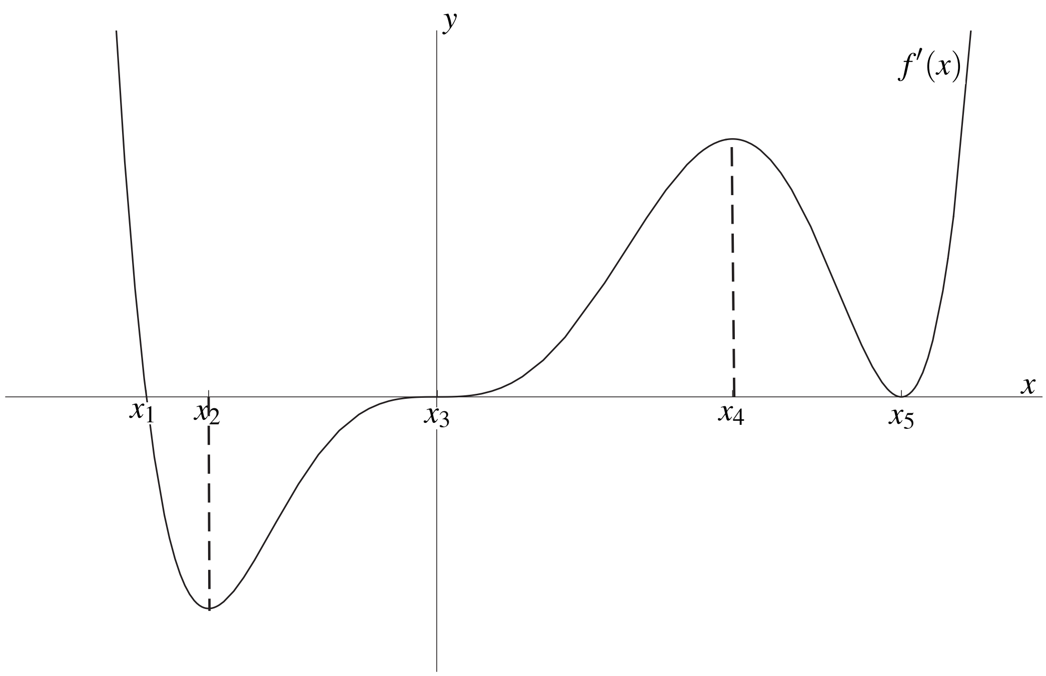
Name: SOLUTIONS

Instructor: ______________________ Section Number: ______________

1. **Do not open this exam until you are told to begin.**
2. This exam has 8 pages including this cover. There are 7 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed two sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to show how you arrived at your solution.
8. Please turn off all cell phones and pagers and remove all headphones.

Problem	Points	Score
1	12	
2	20	
3	18	
4	12	
5	12	
6	14	
7	12	
Total	100	

1. (12 points) The graph below is a plot of $f'(x)$ (the *derivative* of f). Use the graph to answer questions about the function f.



(a) What are the critical points of f? x_1, x_3, x_5

(b) For what value(s) of x does f have a local maximum? x_1

(c) For what value(s) of x does f have a local minimum? x_3

(d) What are the inflection points of f? x_2, x_4, x_5

(e) On what interval(s) is f concave up? $[x_2, x_4], [x_5, \infty)$

(f) If f is a polynomial, what is the minimal degree of f? 7

2. (20 points) Suppose f, g, and h are all differentiable functions of x , $f(x)$ and $g(x)$ are positive for all x, and that a, and b are positive constants. Your answers below will be in terms of f, g, h (and/or their derivatives) and perhaps the constants a or b.

   (a) Find $\dfrac{dy}{dx}$ if $y = f(2) + \ln(f(x^2))$.

$$\frac{dy}{dx} = \frac{f'(x^2)2x}{f(x^2)}$$

   (b) Find $\dfrac{dy}{dx}$ if $y = f(x^a + 2x) + 2^{g(x)}$.

$$\frac{dy}{dx} = f'(x^a + 2x)(ax^{a-1} + 2) + \ln 2(2^{g(x)})g'(x)$$

   (c) Find $\dfrac{dy}{dx}$ if $y = \dfrac{h(bx)}{\cos(x) + 2}$.

$$\frac{dy}{dx} = \frac{bh'(bx)(\cos(x) + 2) + h(bx)\sin(x)}{(\cos(x) + 2)^2}$$

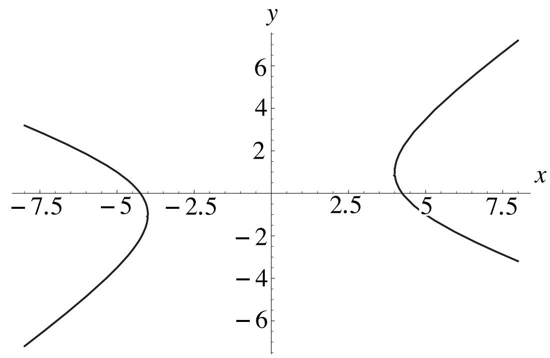
   (d) If $f'(x) = ag(x)$ and $g'(x) = -af(x)$, when is $y = f(x)g(x)$ increasing? [Refer to the instructions above for conditions on f, g and a.] Justify your answer.

$$\begin{aligned} \frac{dy}{dx} &= f'(x)g(x) + f(x)g'(x), \\ &= a(g(s))^2 - a(f(x))^2 = a\left((g^2(x) - f^2(x)\right), \\ &= a\left(g(x) + f(x)\right)\left(g(x) - f(x)\right). \end{aligned}$$

   Thus, since a, f and g are all positive, $\dfrac{dy}{dx} > 0$ when $g(x) > f(x)$.

3. (18 points) Below is a graph of the curve implicitly defined by the equation

$$2y^2 - xy - x^2 = -18.$$



(a) Find a formula for $\frac{dy}{dx}$ as a function of both x and y.

Using implicit differentiation, we have

$$4y\frac{dy}{dx} - y - x\frac{dy}{dx} - 2x = 0, \text{ so } (4y - x)\frac{dy}{dx} = y + 2x$$

which gives $\boxed{\frac{dy}{dx} = \frac{y+2x}{4y-x}.}$

(b) Find the value of $\frac{dy}{dx}$ at the point $(5, -1)$.

$$\frac{dy}{dx}\Big|_{(5,-1)} = \frac{-1+10}{-4-5} = -\frac{9}{9} = -1$$

(c) Find any points (x_0, y_0) where $\frac{dy}{dx}$ is undefined, or give justification why no such points exist.

From above, we know $\frac{dy}{dx}$ is undefined if $4y = x$.

Thus,

$$2y^2 - 4y^2 - 16y^2 = -18,$$

$$\text{so} \quad -18y^2 = -18; \text{ or } y^2 = 1 \text{ which gives } y = \pm 1.$$

If $y = \pm 1$, and $4y = x$, then $x = \pm 4$. $\boxed{\text{The points are } (4, 1) \text{ and } (-4, -1).}$

(d) Find any points (x_0, y_0) where $\frac{dy}{dx} = 0$, or give justification why no such points exist.

The expression for $\frac{dy}{dx}$ will be zero if $y = -2x$, so

$$2(4x^2) + 2x^2 - x^2 = 9x^2 = -18.$$

However, this gives $x^2 = -2$, and there are no real solutions. Thus, the graph has no horizontal tangents, or there are no real values such that $\frac{dy}{dx} = 0$.

4. (12 points) Ellen and Renzo ran the Detroit marathon last weekend. The distance Ellen traveled (in meters) is given by $E(t)$ where t is time measured in seconds since the start of the race. Similarly, the distance in meters Renzo traveled is given by the function $R(t)$. For x measured in meters let $F(x) = R(E^{-1}(x))$. Assume that Ellen moves forward throughout the race–she does not even take a rest!

   (a) What is the practical interpretation of $F(50)$.

   We have

$$F(50) = R(E^{-1}(50))$$

   which gives the distance, in meters, that Renzo has traveled when Ellen has traveled 50 meters.

   (b) After the initial blast of speed from her start, Ellen ran at a constant rate of 5 meters per second for $2 < t < 10$, and she had run a distance of 39 meters after 7 seconds. Renzo wore a device that tracked the distance he had run at one second intervals. The data he collected is summarized in the table below.

t	0	1	2	3	4	5	6	7	8	9	10
$R(t)$	0	10	16	22	28	34	40	46	52	58	64

   Use any of the information above to approximate $F'(39)$.

$$F'(39) = R'(E^{-1}(39))(E^{-1})'(39) = R'(7)(E^{-1})'(3).$$

   From the table, we approximate $R'(7) = 6$, and we know $E^{-1}(39) = 7$ and $E'(7) = 5$. Thus, since

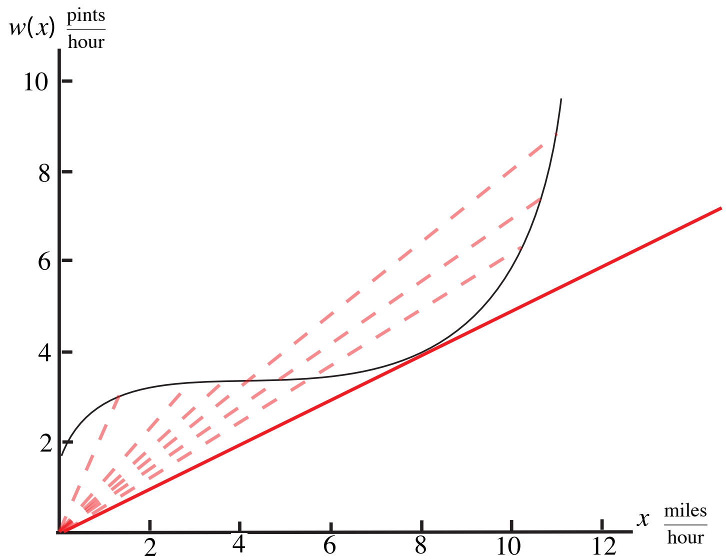
$$(E^{-1})'(39) = \frac{1}{E'(E^{-1}(39))} = \frac{1}{E'(7)} = \frac{1}{5},$$

   we have $\boxed{F'(39) = \frac{6 \text{ meters (for Renzo)}}{5 \text{ meters (for Ellen)}}.}$

   (c) Give a practical interpretation of $F'(39)$.

   Since $F'(39)$ is in $\frac{\text{meters for Renzo}}{\text{meters for Ellen}}$, once Ellen has run 39 meters, if Ellen's distance increases by 1 meter, Renzo will travel approximately 1.2 meters.

5. (12 points) Running a marathon takes a lot of energy. In order to keep up her energy level, Ellen drinks WolverineAid. Let $W(x)$ represent the number of pints of WolverineAid that Ellen must consume per hour when running at a rate of x miles per hour. The graph of $W(x)$ is given below.



(a) Let $C(x)$ represent Ellen's consumption of WolverineAid in pints per mile. How is $C(x)$ related to $W(x)$?

$$C(x) = \frac{W(x)}{x} \frac{\frac{\text{pints}}{\text{hour}}}{\frac{\text{miles}}{\text{hour}}} = \frac{W(x)}{x} \frac{\text{pints}}{\text{mile}}$$

(b) Use calculus to show that $C(x)$ has a critical point at $x = x_0$ when $W'(x_0) = C(x_0)$. (Show your work.)

Assuming $x \neq 0$ and using the quotient rule:

$$C'(x) = \frac{xW'(x) - W(x)}{x^2}$$

If $C'(x) = 0$ at $x = x_0$, then $x_0 W'(x_0) = W(x_0)$. Thus, $W'(x_0) = \dfrac{W(x_0)}{x_0} = C(x_0)$. Since $C'(x_0) = 0$ then x_0 is a critical point of $C(x)$.

(c) From the graph, approximate the pace that Ellen should run in order to get the most efficient use of the WolverineAid. Explain your answer.

To most efficiently use Wolverine Aid, Ellen should minimize $C(x)$. From part (a) we know that $C(x)$ can be represented as the slope of the line connecting the point $(x, W(x))$ and $(0, 0)$. The smallest slope occurs when $x \approx 8$. Also notice that this is indeed a critical point of $C(x)$ as the slope of the line from $(8, 4)$ is the same as the tangent line and consequently $W'(8) = \dfrac{W(8)}{8} = C(8)$.

6. (14 points) Marathons are expensive. The city must use extra police for traffic control, set up aid stations, etc. However, for each runner in the race, the mayor estimates that the city will take in \$150 in revenue from entry fees and the expected items that the runners will buy before, after, and during the race.

(a) The city's cost, in thousands of dollars, for the marathon is given by $C(n) = (n-10)^3 + 1000$, where n is the number, in *thousands*, of runners in the race. What is the Marginal Cost for 5000 runners?

The Marginal Cost for 5000 runners is given by the quantity $C'(5)$ since n is measured in thousands of runners.

$$\begin{aligned} C'(n) &= 3(n-10)^2, \\ C'(5) &= 3(-5)^2 = 75\frac{\text{thousands of dollars}}{\text{thousands of runners}} \end{aligned}$$

The marginal cost for 5000 runners is $75\frac{\text{dollars}}{\text{runner}}$.

(b) If the city has 5000 entrants for the race, would they want to increase or decrease the number of entrants (or stay at 5000). Explain.

Marginal profit is equal to marginal revenue (MR) minus marginal cost (MC). From part (a) the marginal cost is $75\frac{\text{dollars}}{\text{runner}}$ when there are 5000 entrants. Additionally, from the information given we know that the marginal revenue is always $150\frac{\text{dollars}}{\text{runner}}$. Thus, the Marginal Profit when $n = 5$ is

$$MR - MC = 150 - 75 = 75\frac{\text{dollars}}{\text{runner}}.$$

Consequently, by increasing the number of entrants from 5000 to 5001 the cities profit will increase by approximately \$75 and should thus the city should increase the number of entrants.

(c) What number of runners will maximize the city's profit? [You may use your calculator for portions of this problem, but be sure to justify your answer.]

Critical points of the profit function, occur when Marginal Profit=0. (Note, there are no places where Marginal Profit is undefined.) Marginal Profit is zero when $MC = MR$. Since $MR = 150$ for all n, and from part (a) $MC = C'(n) = 3(n-10)^2$, critical points occur when:

$$3(n-10)^2 = 150.$$

Solving for n we find that critical points occur at $n = 10 + \sqrt{50} \approx 17.071$ thousands of entrants and $n = 10 - \sqrt{50} \approx 2.92$ thousands of entrants. To determine the global max find the profit at each critical point as well as the behavior of the profit as $n \to \infty$ and at $n = 0$. The profit in *thousands* of dollars is given by $\pi(n) = 150n - (n-10)^3 - 1000$ where n is in thousands of people. Using a calculator,

$$\begin{aligned} &\pi(0) = 0; && \boxed{\pi(17.071) = 1207.10678} \\ &\pi(2.92) = -207.10508; && \lim_{n\to\infty} \pi(n) = -\infty \end{aligned}$$

Thus the maximum profit of \$1,207,106.78 occurs when there are 17,071 entrants in the marathon.

7. (12 points) The flux F, in millilitres per second, measures how fast blood flows along a blood vessel. Poiseuille's Law states that the flux is proportional to the fourth power of the radius, R, of the blood vessel, measured in millimeters. In other words $F = kR^4$ for some positive constant k.

(a) Find a linear approximation for F as a function of R near $R = 0.5$. (Leave your answer in terms of k).

The linear approximation near $R = 0.5$ is given by $F(R) \approx F(0.5) + F'(0.5)(R - 0.5)$.

$$F(0.5) = k(0.5)^4 = \tfrac{k}{16},$$
$$F'(R) = 4kR^3$$

$$F'(0.5) = 4k(0.5)^3 = \tfrac{k}{2}$$

Thus for R near 0.5, $F(R)$ is given by:

$$\boxed{F(R) \approx \tfrac{k}{16} + \tfrac{k}{2}(R - 0.5)}$$

(b) A partially clogged artery can be expanded by an operation called an angioplasty, which widens the artery to increase the flow of blood. If the initial radius of the artery was 0.5mm, use your approximation from part (a) to approximate the flux when the radius is increased by 0.1mm.

To approximate when the radius increases by 0.1mm from 0.5mm we can evaluate the linear approximation from (a) at 0.6 giving:

$$\begin{aligned} F(0.6) &\approx \frac{k}{16} + \frac{k}{2}(0.6 - 0.5), \\ &= \frac{k}{16} + \frac{k}{20}, \\ &= \frac{9k}{80} = \boxed{(0.1125)\text{k} \ \ \text{millilitres per second}}. \end{aligned}$$

(c) Is the answer found in part (b) an under- or over-approximation? Justify your answer.

In order to determine if this is an over or an under approximation we use the second derivative:

$$F''(R) = 12kR^2$$

Since $k > 0$, $F''(R)$ is always positive, and the function is always concave up. Thus the linear approximation will always be an *under approximation*