Math 115 – Final Exam

December 18, 2006

Name: ______________________________

Instructor: ______________________ Section Number: ______________

1. **Do not open this exam until you are told to begin.**
2. This exam has 8 pages including this cover. There are 7 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed two sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to show how you arrived at your solution.
8. Please turn off all cell phones and pagers and remove all headphones.

Problem	Points	Score
1	12	
2	24	
3	12	
4	10	
5	10	
6	16	
7	16	
Total	100	

1. (3 points each) In each of the following, circle **one** of the answers (A)-(E). No explanation necessary.

   (a) If f is differentiable for all x and has a local maximum at $x = 3$, then which of the following **must** be true?

   I. $f'(3) = 0$
   II. $f''(3) < 0$
   III. f is continuous at $x = 3$

   (A) I only (B) II only (C) I and II only

   $\boxed{\text{(D) I and III only}}$ (E) I, II, and III

   (b) If f and g are differentiable, $h(x) = f(x) - g(x)$, and $h(x)$ has a local maximum value at $x = 3$, then

   (A) $f'(x) > g'(x)$ $\boxed{\text{(B)} f'(3) = g'(3)}$ (C) $f'(3) < g'(3)$

   (D) $f(x)$ has a local maximum value at $x = 3$

   (E) $g(x)$ has a local minimum value at $x = 3$

   (c) Let $f(x) = \dfrac{\sin(x)}{e^x}$ for $x > 0$. When the minimum value of $f(x)$ occurs, then

   (A) $\sin(x) = 0$ (B) $\cos(x) = 0$ $\boxed{\text{(C) } \cos(x) = \sin(x)}$

   (D) $\cos(x) = -\sin(x)$ (E) $f(x)$ does not have any extreme values on the interval $[0, \infty)$

   (d) The graph of $y = x + \dfrac{1}{x}$ is both increasing and concave down on the interval

   $\boxed{\text{(A) } (-\infty, -1)}$ (B) $(-1, 0)$ (C) $(0, 1)$

   (D) $(1, \infty)$ (E) never

2. (4 points each) Suppose that f, g and h are continuous and differentiable functions such that $f'(x) = g(x)$ and **ALL** of the following conditions are also true:

$$\int_0^5 f(x)dx = -2, \qquad \int_5^{10} g(x)dx = 2, \qquad \int_0^5 g(x)dx = 15,$$

$$f(0) = 7, \qquad h(x) = g(x-5)$$

For parts (a)-(f), find the numerical value indicated. If insufficient information is given to answer the question indicate "Insufficient information".

(a) $\int_0^5 f(0)g(x)dx = f(0)\int_0^5 g(x)dx = 7(15) = 105$

(b) $f(10) = \int_0^{10} g(x)dx + f(0) = \int_0^5 g(x)dx + \int_5^{10} g(x)dx + f(0) = 15 + 2 + 7 = 24$

(c) $\int_0^5 |f(x)|\, dx =$ Insufficient Information

(d) $\int_0^5 \left(3f(0) - \frac{g(x)}{5}\right) dx = 3f(0)\int_0^5 dx - \frac{1}{5}\int_0^5 g(x)dx = 3(7)(5) - \frac{1}{5}(15) = 102$

(e) $\int_0^5 \frac{1}{g(x)}dx =$ Insufficient Information

(f) $\int_5^{10} h(x)dx = \int_0^5 g(x)dx = 15$

3. (12 points) Consider the family of cubics of the form

$$f(x) = ax^3 + bx + c$$

with a, b, and c non-zero constants.

(a) (2 points) Using the function $f(x) = ax^3 + bx + c$ as given above, write the **limit definition** of the derivative function, $f'(x)$. (No need to expand or simplify–just apply the definition to this function, using proper notation.)

$$f'(x) = \lim_{h \to 0} \frac{a(x+h)^3 + b(x+h) + c - ax^3 - bx - c}{h}$$

(b) (6 points) Under what conditions, if any, on a, b, and c will f have local extrema (i.e., maxima/minima)?

Since f is a polynomial, so critical points only occur when $f'(x) = 0$.

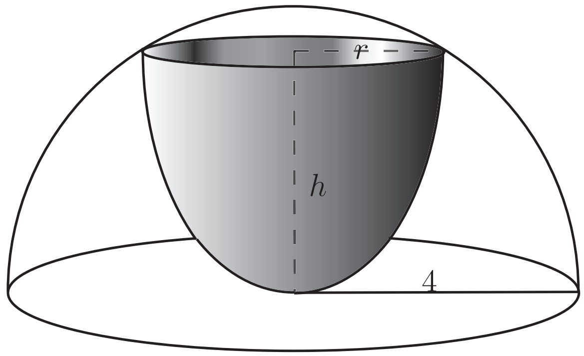
$$f'(x) = 3ax^2 + b$$

$f'(x) = 0$ if there are real values of x such that $2ax^2 + b = 0$, this only occurs if $-\frac{b}{3a} > 0$ (or if a and b are opposite signs). If this conditions holds for a and b then there are two critical points at $x = \pm\sqrt{-\frac{b}{3a}}$. To check that these are indeed extrema, one could take the second derivative $f''(x) = 6ax$, and then $f''(\pm\sqrt{-\frac{b}{3a}}) = \pm 6a\sqrt{-\frac{b}{3a}}$. Thus indeed, if a and b are opposite signs, then the second derivative is either positive or negative (since a and b are non-zero) at the critical points, and consequently each critical point is indeed an local extrema under this condition.

(c) (4 points) Under what conditions, if any, on a, b, and c will f have inflection point(s)?

Possible inflection points occur when $f''(x) = 0$. The second derivative of $f(x)$ (as calculated above) is $f''(x) = 6ax$. Since a is non-zero then the only *possible* inflection point occurs when $x = 0$. To check that this is indeed an inflection point we must check the concavity does indeed change at $x = 0$. This is indeed the case, for if $a > 0$ then $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$. Consequently $x = 0$ is an inflection point for ANY values of a, b, and c.

4. (10 points) Suppose a paraboloid cup is inscribed in a hemisphere of radius 4 inches. The volume of the paraboloid is given by $\frac{1}{2}\pi r^2 h$. For what values of the parameters r and h is the volume of the cup maximized?



One can envision r and h being the coordinates of a point on a circle of radius 4, thus r and h must be related by:

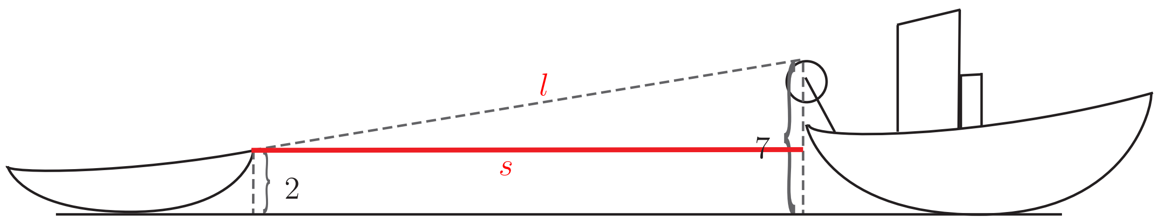
$$r^2 = 16 - h^2.$$

Using this relationship and the given formula for the volume of the paraboloid, V, ($V = \frac{1}{2}\pi r^2 h$) we can write V in terms of h only by replacing r^2. Namely,

$$V = \frac{1}{2}\pi(16 - h^2)h\,.$$

Differentiating V with respect to h gives, $dV/dh = \frac{1}{2}\pi(16 - 3h^2)$. Critical points occur at values of h when $dV/dh = 0$, such values of h are $h = \pm\frac{4}{\sqrt{3}}$. Because h represents a height and since it must be less then 4 (since it is inscribed in the sphere), then we only are interested in $0 \leq h \leq 4$. Clearly if $h = 4$ of $h = 0$ then the volume is 0. The only critical point between 0 and 4 is the positive critical point $h = \frac{4}{\sqrt{3}}$, substituting back into the equation for the volume we see that when $h = \frac{4}{\sqrt{3}}$, the volume is $V = \frac{1}{2}\pi\left(16 - \left(\frac{4}{\sqrt{3}}\right)^2\right)\left(\frac{4}{\sqrt{3}}\right) = \frac{64\pi}{3\sqrt{3}}$. Consequently, since this volume is positive, it must be the maximum volume. We can find the radius at this value of h using $r^2 = 16 - h^2$ to get $r = 4\sqrt{\frac{2}{3}}$. Consequently, the volume is maximized when $h = \frac{4}{\sqrt{3}}$ inches and $r = 4\sqrt{\frac{2}{3}}$ inches.

5. (10 points) A small boat has run out of gas. A cable is attached to the front of the boat 2 meters above the water. The other end of the cable is attached to a wheel of radius 0.5 meters sitting on the back of a tugboat. The top of the wheel is 7 meters above the water, and turns at a constant rate of 1 revolution per second. [See the figure below–not drawn to scale.]



(a) At what rate is the length of the cable between the two boats changing?

Let n represents the number of rotations of the wheel and l the length of the cable between the two boats. We are only interested in the rate at which l is changing, and l decreases by $2\pi(.5) = \pi$ meters per rotation. Since the wheel is being turned at 1 revolution per second then l is decreasing by π meters per second. In other words $dl/dt = -\pi \frac{\text{meters}}{\text{second}}$.

(b) How fast is the small boat being pulled forward when it is 10 meters away from the tugboat?

Let s represent the distance between the two boats (see the diagram). Using the Pythagorean Theorem, $s^2 + 25 = l^2$. Solving for s,

$$s = \sqrt{l^2 - 25}.$$

There is no ambiguity here because s and l are both positive distances. Differentiating this relation with respect to t gives:

$$\frac{ds}{dt} = \frac{l}{\sqrt{l^2 - 25}} \frac{dl}{dt}.$$

From the information in part (a) we know that $dl/dt = -\pi \frac{\text{m}}{\text{s}}$. Additionally we want to know ds/dt when $s = 10$ meters, using again that $l^2 = s^2 + 25$ we see that $l = 5\sqrt{5}$ when $s = 10$. Consequently,

$$\frac{ds}{dt} = \frac{5\sqrt{5}}{\sqrt{(5\sqrt{5})^2 - 25}}(-\pi) = -\frac{\sqrt{5}}{2}\pi \frac{\text{m}}{\text{s}}.$$

6. Suppose $H(c)$ gives the average temperature, in degrees, that can be maintained in Oscar's apartment during the month of December as a function of the cost of the heating bill, c, in dollars. In complete sentences, give a practical interpretation of the following:

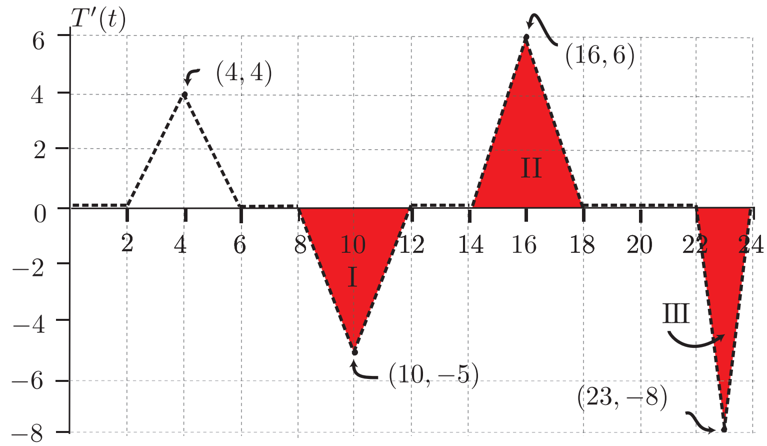
   (a) (3 points) $H(50) = 65$

   The practical interpretation of $H(50) = 65$ is that if Oscar's heating bill costs \$50.00 in December then he is able to maintain an average temperature during that month of 65°.

   (b) (3 points) $H'(50) = 2$

   The practical interpretation of $H'(50) = 2$ is that if Oscar's December heating bill increases from \$50.00 to \$51.00, then the average temperature he can maintain during that month will change from 65° to approximately 67°.

   Suppose $T(t)$ gives the temperature in Oscar's apartment on December 18th in °F as a function of the time, t, in hours since 12:00 midnight. Below is a graph of $T'(t)$: (**NOTE: the graph is of $T'(t)$.**)



   (c) (6 points) When Oscar gets home from work at 6 pm the temperature in his apartment is 67 degrees. What was the temperature when he left for work at 8 am?

   Using the Fundamental Theorem of Calculus we know that

$$\int_8^{18} T'(t)dt = T(18) - T(8).$$

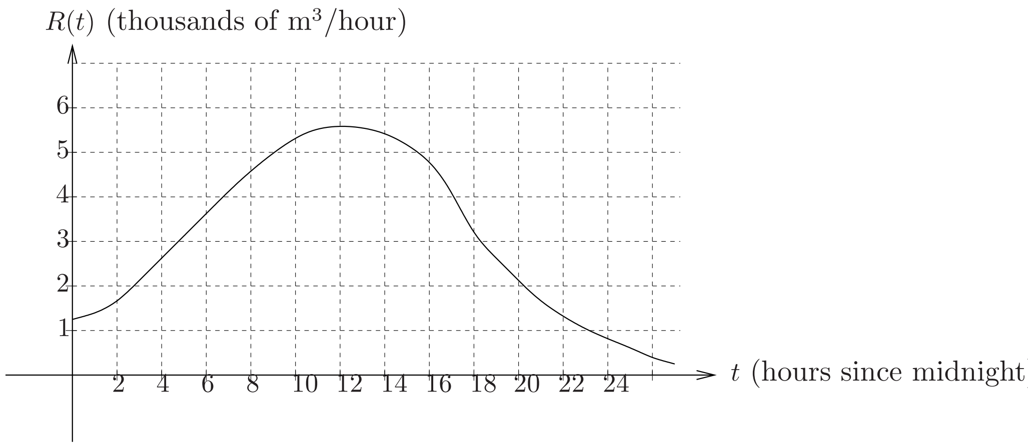
   We are given that $T(18) = 67°$. Computing the areas I and II on the graph of T' we know that the $\int_8^{18} T'(t)dt =$-II+I$= -10 + 12 = 2$ Consequently, $2 = 67° - T(8)$. Thus the temperature at 8 am is $T(8) = 65°$.

   (d) (4 points) If the temperature at 6 pm is 67 degrees, what is the minimum temperature in the apartment on December 18th?

   When $T'(t) > 0$ ($T'(t) < 0$) then T is increasing (decreasing) and when $T'(x)$ is 0 on an interval the temperature is constant. Thus for $12 < t < 2$, $6 < t < 8$, $12 < t < 14$ and $18 < t < 22$ the temperature is not changing, for $2 < t < 6$ and $14 < t < 18$ the temperature is increasing, while it is decreasing for $8 < t < 12$ and $22 < t < 24$. Consequently the minimum temperature either occurs at $t = 0, t = 12$ or $t = 24$. Using the Fundamental Theorem of Calculus, and our knowledge of interpreting integrals as areas we have, $T(12) = T(18) - \int_{12}^{18} T'(t)dt = 67 - II = 67 - 12 = 55°$ and $T(24) = T(18) + \int_{18}^{24} T'(t)dt = 67 - III = 67 - 8 = 59°$, and similarly $T(0) = 59°$. Consequently, the minimum temperature occurs anywhere between noon and 2pm. This minimum temperature is 55°F.

7. Marie is already tired of winter. She is dreaming of her grandparents' farm and days rafting on the river near the farm. Not all days on the river are beautiful, though. One summer a storm dumped about a year's worth of rainfall on the area in a couple of days. A man-made lake held back by a dam near the farm rose as the swollen rivers rushed toward the lake.

   The graph below gives the rate R, in thousands of cubic meters per hour, that water was entering the lake during that day as a function of t, in hours since midnight. The volume of the lake at midnight was 400,000 cubic meters. The maximum volume that can be held by the dam is 460,000 cubic meters. Due to an oversight, the floodgates of the dam were kept closed until 6:00 a.m when they were opened to full capacity. The gates allowed water to leave the lake at a constant rate of 2000 cubic meters per hour.



   (a) (4 points) Approximate the volume of the lake when the floodgates were opened. Show your reasoning.

   Suppose $V(t)$ is the total volume at time t. Before the floodgates open $R(t) = V'(t)$. The volume at $t = 0$ is $400,000\text{m}^3$, thus $V(0) = 400$. The volume when the floodgates open is given by $V(6)$. By the fundamental theorem $V(6) = \int_0^6 R(t)dt + V(0) \approx 13 + 400 = 413$ thousands of m^3. The integral was approximated by counting approximately 6.5 squares (each of which has an area of 2) in the region below the graph of $R(t)$, above the t axis, and between $t = 0$ and $t = 6$.

   (b) (4 points) When did the lake reach its highest volume? Explain

   After the floodgates open the rate at which the volume is changing is given by $dV/dt =$(Rate in)-(Rate out)$=$ $(R(t) - 2)\frac{\text{thousands of m}^3}{\text{s}}$. Since at time $t = 6$ $R(t) > 2$ then $dV/dt > 0$ and thus the volume is increasing even though water is being let out. The volume continues to increase until $dV/dt = 0$ which occurs when $R(t) = 2$ (for $t > 6$). This happens at approximately $t = 20$ (or 8 pm). After that $R(t) < 2$ and $dV/dt < 0$ so the volume is decreasing. Thus the highest volume of the lake occurs at about 8 pm.

   (c) (5 points) Approximately what was the highest volume of the lake on that day? Explain.

   Since the highest volume occurs when $t = 20$ we can use the fundemental theorem of Calculus and the fact that $V'(t) = R(t)$ for $t < 6$ and $V'(t) = R(t) - 2$ for $6 \le t \le 20$, namely,

$$\begin{aligned} V(20) &= \int_0^{20} V'(t)dt + V(0) = \int_0^6 R(t)dt + \int_6^{20} (R(t) - 2)dt + V(0) \\ &= \int_0^{20} R(t)dt - 2(14) + 400 = 76 - 28 + 400 = 448 \text{ thousands of m}^3 \end{aligned}$$

   Here the integral of $R(t)$ was approximated by counting approximately 38 squares (each of which has an area of 2) in the region below the graph of $R(t)$, above the t axis between $t = 0$ and $t = 20$.

(d) (3 points) At what rate was the volume of the lake changing at 6:00 pm?.

The rate that the volume was changing at a given time after 6 am is given by $V'(t) =$(Rate in) - (Rate out)$=R(t) - 2$. 6:00 pm corresponds to $t = 18$. If we approximate the value of $R(18)$ by looking at the graph we conclude that $V'(18) = 3.2 - 2 = 1.2 \frac{\text{thousands of m}^3}{\text{s}}$. Thus at 6:00 pm the volume of water in the lake is increasing at a rate of $1.2 \frac{\text{thousands of m}^3}{\text{s}}$ (or $1200\text{m}^3/\text{s}$).