MATH 115 – FIRST MIDTERM EXAM

Solutions

October 9, 2007

NAME: ______________________________

INSTRUCTOR: ______________________ SECTION NUMBER: ______________

1. **Do not open this exam until you are told to begin.**
2. This exam has 9 pages including this cover. There are 9 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed two sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to show how you arrived at your solution.
8. Please turn off all cell phones and pagers and remove all headphones.

PROBLEM	POINTS	SCORE
1	18	
2	14	
3	13	
4	7	
5	8	
6	12	
7	6	
8	14	
9	8	
TOTAL	100	

1. (3 points each. No partial credit.) The questions on this page are True or False. They do not require an explanation. For each question, circle your choice for the correct answer. Only answer True when the statement is **ALWAYS** True.

(a) The function $f(x) = \dfrac{e^x}{x^2 - 1}$ is continuous on $[2,\ 5]$.

$\boxed{\text{True}}$ False

(b) Suppose g is a differentiable function on $(-1,\ 1)$ with $g(1) < 0$ and $g'(x) > 0$ for x in $(-1,\ 1)$, then $g(x)$ has a zero on the interval $[-1,\ 1]$.

True $\boxed{\text{False}}$

(c) If $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$ then f is continuous at $x = 0$.

True $\boxed{\text{False}}$

(d) If $x > 0$ and $e^{xy-2} = x^2$, then $y = \dfrac{2}{x}(1 + \ln\ x)$.

$\boxed{\text{True}}$ False

(e) A function that is continuous on $[a,\ b]$ is always differentiable on $[a,\ b]$.

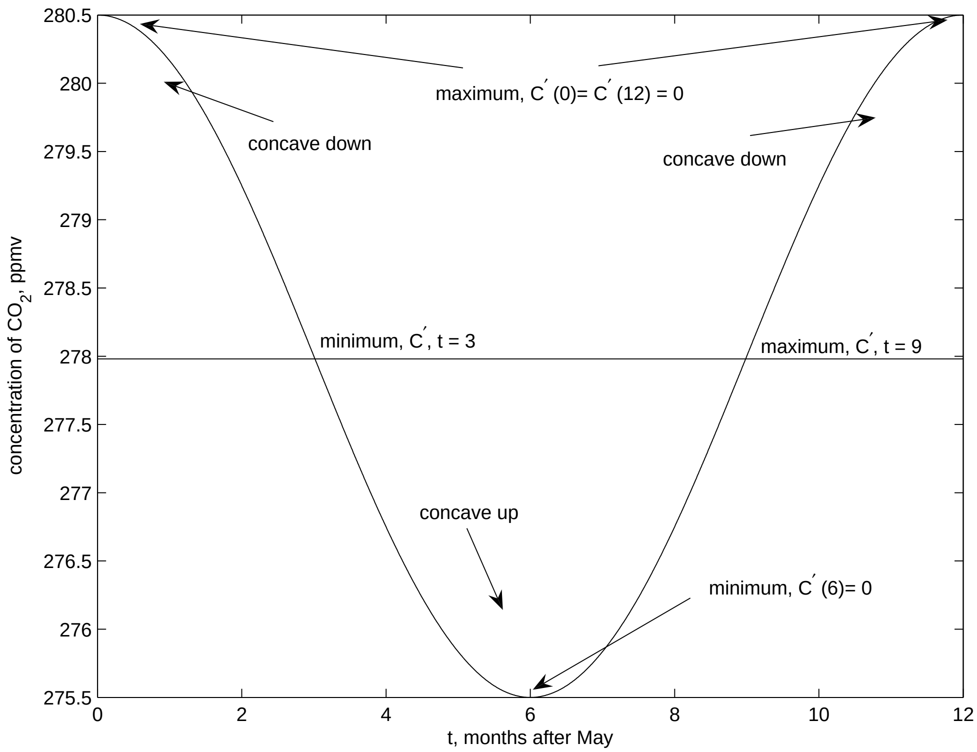
True $\boxed{\text{False}}$

(f) If $f'(a) = 0$ and $f'(x) > 0$ for $x < a$, then $f''(x) > 0$ for $x < a$.

True $\boxed{\text{False}}$

2. Carbon dioxide concentrations in the atmosphere fluctuate seasonally due to the amount of CO_2 taken up by plants. Scientists believe that for thousands of years prior to the industrial era (circa 1800) the cycle fluctuated each year around an average concentration of about 278 ppmv (that is, 278 molecules of CO_2 for every one million molecules of air). [1]

   (a) (4 points) Assuming 278 ppmv as the average and a total fluctuation of 5 ppmv between the high and low concentrations, draw a graph of $C(t)$, with t in months, which models the level of CO_2 concentration during the years preceding the industrial era. Let $t = 0$ represent the beginning of the month of May when the concentration is at the highest. Carefully label important features on your graph.



   (b) (5 points) Determine a trigonometric formula for $C(t)$.

$$C(t) = 278 + \frac{5}{2}\cos\left(\frac{\pi}{6}\,t\right)$$

   (c) (2 points) What is the amplitude of the function C? $\frac{5}{2}$

   (d) (3 points) At the beginning of which month during the year is the concentration level decreasing fastest? August

[1] See http://en.wikipedia.org/wiki/Global_warming

3. Dr. Charles Keeling began measuring carbon dioxide in the atmosphere on a continuous basis in 1958. At that time, Dr. Keeling found that the mean concentration level was approximately 315 ppmv. Currently, the level is approximately 385 ppmv. [2]

   (a) (4 points) Assuming that the mean concentration has been growing linearly from 1958 to 2007, find a formula for $L(t)$, the mean concentration level of CO_2, with t in years since 1958.

   $$L(t) = 315 + \frac{385 - 315}{2007 - 1958}\, t = 315 + 1.42857\, t,$$

   where t is the number of years since 1958.

   (b) (5 points) If instead, the mean concentration has been growing at an exponential rate, find an exponential function, $E(t)$, to model the mean concentration level of CO_2 in the environment t years after 1958.

   Let t be the number of years from 1958.

   $$E(t) = c\, e^{a\,t} \text{ at } t = 0, \quad E(0) = 315, \quad \text{therefore} \quad c = 315.$$

   Also, $E(49) = 385$, so we have

   $$385 = 315\, e^{49a}.$$

   Taking natural logs:

   $$\ln \frac{385}{315} = 49a, \quad \text{therefore} \quad a = \frac{\ln \frac{385}{315}}{49}$$

   and

   $$E(t) = 315\, e^{0.004095\, t}.$$

   Students may opt to take $E(t) = p\, b^t$, and $E(0) = 315$ so $p = 315$. Then $E(49) = 385$ gives

   $$385 = 315\, b^{49}. \quad \text{Rearranging and solving for } b \text{ gives} \quad b = \left(\frac{385}{315}\right)^{1/49},$$

   so $E(t) = 315(1.0041)^t$. Note this is equivalent to the answer given above.

   (c) Future CO_2 levels are expected to rise due to burning of fossil fuels and land-use changes. The rate of this increase will depend on uncertain economic, sociological, technological, and natural developments. The IPCC Special Report on Emissions Scenarios gives a wide range of CO_2 scenarios by the year 2100. Use your functions from parts (a) and (b) to predict the concentration of CO_2 in 2100.

   1 point taken off if the units are not included.

   (i) (2 points) Prediction if growth is linear:

   $$L(142) = 517.857 \quad \text{ppmv}$$

   (ii) (2 points) Prediction if growth is exponential:

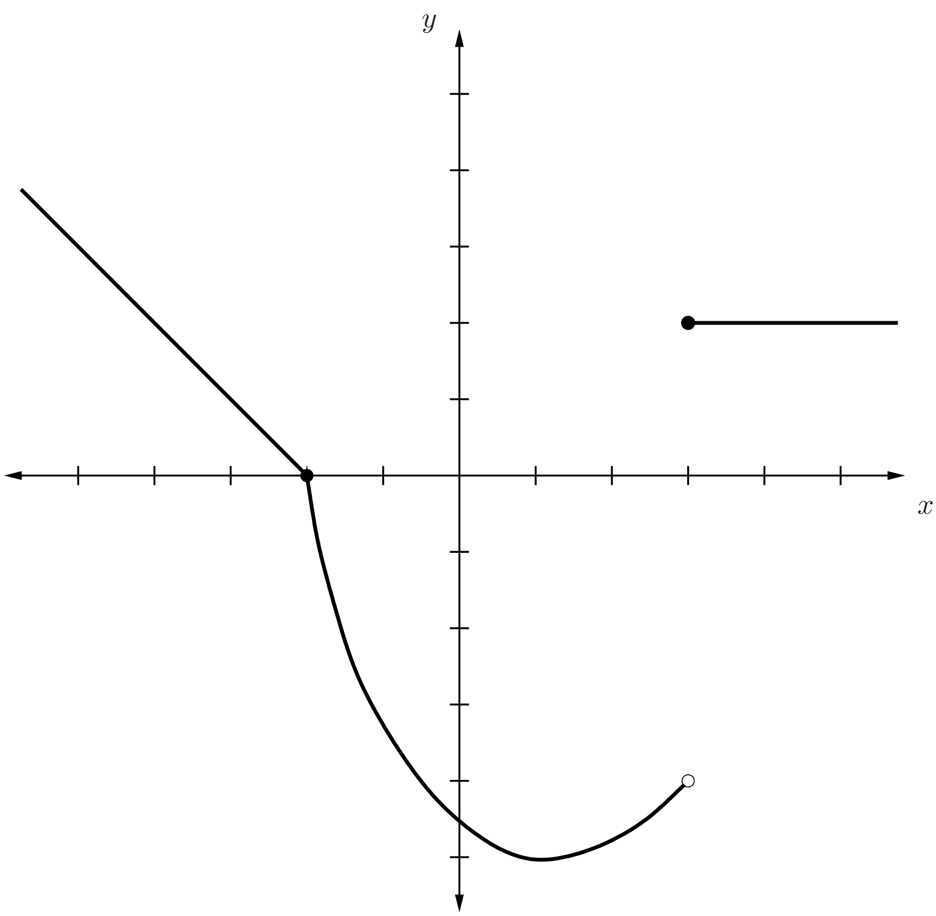
   $$E(142) = 563.467 \quad \text{ppmv}$$

[2] See `http://earthguide.ucsd.edu/globalchange/keeling_curve/01.html` and `http://en.wikipedia.org/wiki/Carbon_dioxide`

4. (7 points) On the axes below, sketch a graph of a single function, g, with all of the following properties.

- $g(-2) = 0$
- $g'(x) = -1$ for $x < -2$
- $\lim_{x \to -2^-} g(x) = \lim_{x \to -2^+} g(x)$
- $g''(x) > 0$ for $-2 < x < 0$
- $g'(1) = 0$
- $\lim_{x \to 3^-} g(x) = -4$
- $g(x) = 2$ for $x \geq 3$

Note: Answers are not unique. One example of a graph which has the given properties is below.



5. (a) (4 points) Let the function f be defined as follows:

$$f(x) = \begin{cases} 2^p\ (x-1) & \text{for } \quad x > 2 \\ x^2 & \text{for } 0 \le x \le 2 \\ \cos(x^2) + k & \text{for } \quad x < 0 \end{cases}$$

Find the values of p and k so that f is a continuous function.

For p we need to make $2^p\ (x-1) = x^2$ at $x = 2$, *i.e.*,

$$2^p = 4, \quad \text{and thus} \quad p = 2.$$

$p =$ 2 ________________

For k we need to make $\cos(x^2) + k = x^2$ at $x = 0$, *i.e.*,

$$1 + k = 0, \quad \text{and thus} \quad k = -1.$$

$k =$ -1 ________________

(b) (4 points) Using $f(x)$ as determined in part (a) and $g(x)$ given by:

$$g(x) = \begin{cases} \dfrac{x^3}{3} & \text{for } x \ge 3 \\ |x| & \text{for } x < 3 \end{cases}$$

find

(i) $\lim_{x \to 3^+} f(x)g(x)$

This is the limit from the RIGHT. Therefore

$$\lim_{x \to 3^+} f(x)g(x) = \lim_{x \to 3^+} \frac{x^3}{3}\ 4\ (x-1) = 72$$

(ii) $\lim_{x \to 3^-} f(x)g(x)$

This is the limit from the LEFT. Therefore

$$\lim_{x \to 3^-} f(x)g(x) = \lim_{x \to 3^+} |x|\ 4\ (x-1) = 24$$

6. Adrian wants to conserve energy. She learned that as a nation we spend approximately one quarter of our electricity on lighting. In our homes, the amount of electricity used for lighting can be reduced by using fluorescent light bulbs. If Adrian's monthly electricity bill, L in dollars, is a function of the percent, p, of fluorescent bulbs in her home, give the practical interpretation of the each of following—*i.e.*, give the meaning of the expression or statement as you would explain it to a person who knows no mathematics.

   (a) (4 points) In the context of this problem, $L(0) = 100$ indicates...

   When Adrian has no fluorescent light bulbs in her home, her monthly electric bill is $100.

   (b) (4 points) In the context of this problem, $L'(25) = -2$ indicates...

   If Adrian increases the percentage of fluorescent light bulbs in her house from 25% to 26% her monthly electric bill will decrease by approximately $2.

   (c) (4 points) What does $L^{-1}(75)$ stand for?

   The notation $L^{-1}(75)$ stands for the percentage of fluorescent light bulbs Adrian needs to have in her house so that her monthly bill is $75.

7. (6 points) In the wee hours of October 1st, the Michigan governor and state legislators came up with a new function called "Save Our State," denoted SOS(x). The function is defined for all values and differentiable everywhere. (We all hope the slope is *positive*!) Use the **limit** definition of the derivative to define $f'(2)$ in terms of the function SOS, if

$$f(t) = \text{SOS}(2t^2).$$

$$f'(2) = \lim_{h \to 0} \frac{\text{SOS}(2\,(2+h)^2) - \text{SOS}(8)}{h}$$

8. David is living on the 37th floor of a fancy building. He wants to get rid of an ancient (very energy inefficient) refrigerator that was in the building before alterations were made to the apartment. The box will not fit through the new doors of the apartment, so the refrigerator must be pushed down a rather rickety ramp out the window. The ramp is 350 feet long. Below is a table showing the distance from the window along the ramp at given times:

time (seconds)	0	2	4	6	8	10	12	14	16
distance from window (feet)	0	3.9	18.6	43.1	79.4	122.5	174.6	240.1	313.6

Suppose $s(t) = d$ is the distance from the window, in feet, as a function of time, t, in seconds.

(a) (3 points) Compute the average velocity of the refrigerator over the time interval $4 \leq t \leq 12$.

The average velocity over the time interval $4 \leq t \leq 12 = \dfrac{174.6 - 18.6}{12 - 4} = 19.5$ ft / sec

(b) (4 points) Approximate the instantaneous velocity of the refrigerator when $t = 8$ seconds.

Av. vel. over $6 \leq t \leq 8 = \dfrac{79.4 - 43.1}{2} = 18.15$ ft / sec..

Av. vel. over $8 \leq t \leq 10 = \dfrac{122.5 - 79.4}{2} = 21.55$ ft / sec.

Or, we could average those velocities, giving the approximate instantaneous velocity at $t = 8$ as 19.85 ft / sec.

Note: any of the three approximations were accepted.

(c) (3 points) Approximately where will the refrigerator be after 18 seconds? Justify your answer.

The refrigerator will be on the sidewalk.

This is because the approximation of the velocity at 16 seconds is: $\dfrac{313.6 - 240.1}{2} = 36.75$ ft / sec.

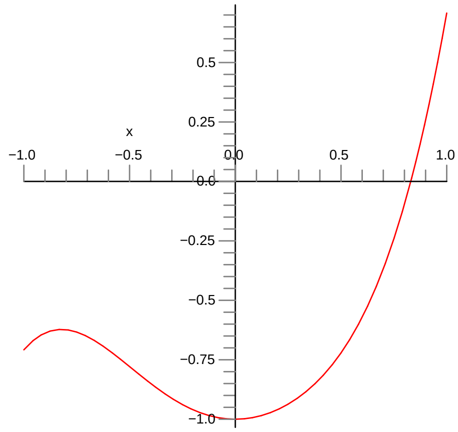
which would mean that in 2 more seconds it would travel 73.5 ft,

but it is only 36.4 ft to the ground.

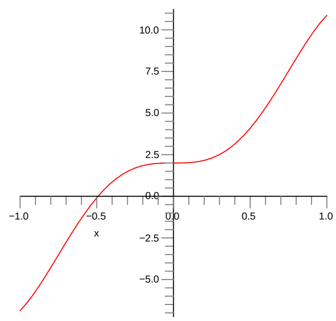
(d) (4 points) Based upon the information in the table, does s appear to be concave up or concave down at $t = 8$? Justify your answer.

The velocity appears to be increasing at $t = 8$, because the average velocity for $6 \leq t \leq 8$ is less than the average velocity for $8 \leq t \leq 10$. Thus, if s' is increasing, s'' is positive, and the graph would be concave up.

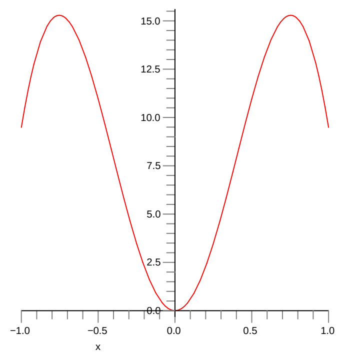
9. (8 points) The graph of $y = g(x)$ is given by the figure below.



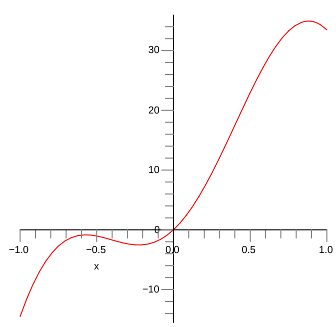
The graphs of the following functions include the graphs of g' and g'' and two additional graphs. On the lines beneath the appropriate graphs, **clearly** identify the graphs of g' and g''. (No explanation necessary.) For each of the other graphs, give *any* feature of the graph (*e.g.*, its behavior on an interval or at a point) which disqualifies that graph as a candidate for the derivative of g. (No need to list *all* reasons!)



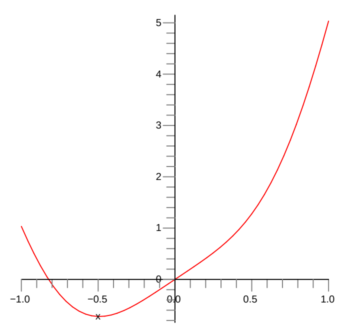
This is g''



Not g'—On (-0.85,0) g is decreasing, so g' must be negative. This graph is positive on (-0.85,0).



Not g'—On (-1,-0.85) g is increasing, so g' must be positive. This graph is negative on (-1,-0.85).



This is g'