MATH 115 – SECOND MIDTERM EXAM

November 13, 2007

NAME: ______________________________

INSTRUCTOR: ____________________ SECTION NUMBER: ____________

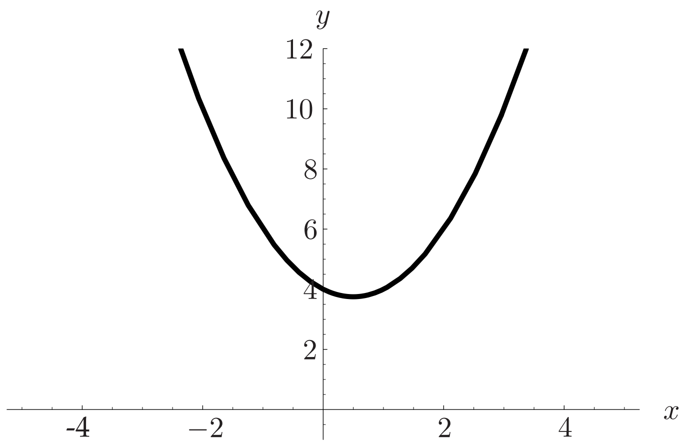
1. **Do not open this exam until you are told to begin.**
2. This exam has 8 pages including this cover. There are 8 questions.
3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.
4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
6. You may use your calculator. You are also allowed two sides of a 3 by 5 notecard.
7. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to show how you arrived at your solution.
8. Please turn off all cell phones and pagers and remove all headphones.

PROBLEM	POINTS	SCORE
1	10	
2	20	
3	14	
4	10	
5	12	
6	5	
7	12	
8	17	
TOTAL	100	

1. (10 points) Using the graph in the figure below to help you, find the equations of all lines through the origin and tangent to the parabola

$$y = x^2 - x + 4.$$

[Show all relevant work—graphical approximations are not sufficient. Please circle your answers.]



The equation of any line,l, through the origin is $l = mx$ since the y-intercept is zero. A line that is tangent to the curve must have the same slope as the original curve at a given point, so

$$y' = 2x - 1 = m$$

.

Therefore, at a point $x = a$ the equations of the lines tangent to the curve must satisfy

$$\underbrace{a^2 - a + 4}_{y} = \underbrace{(2a-1)}_{m} \cdot \underbrace{a}_{x}$$

which implies $a^2 = 4$, so $a = \pm 2$.

If $a = x = 2, y = 6$, and $m = 3$, so the equation of the line is $\boxed{y = 3x}$.

If $a = x = -2, y = 10$, and $m = -5$, so the equation of the line is $\boxed{y = -5x}$.

2. (20 points) Suppose f and g are differentiable functions with the following values:

$$f(0) = 3, \quad f'(0) = 4, \quad g(0) = -1, \quad \text{and} \quad g'(0) = 2.$$

Show your work on the following:

(a) Find $h'(0)$ given $h(x) = \dfrac{g(x)}{f(x)}$.

$h'(x) = \dfrac{g'(x)f(x) - g(x)f'(x)}{f^2(x)}$ thus $h'(0) = \dfrac{(2)(3) - (-1)(4)}{9} = \dfrac{10}{9}$

(b) i. Find $k'(0)$ given $k(x) = (g(x))^2 f(x)$.

$k'(x) = 2g(x)g'(x)f(x) + (g(x))^2 f'(x)$.

Thus $k'(0) = 2(-1)(2)(3) + (-1)^2 4 = -8$

ii. Determine the local linearization of $k(x)$ near $x = 0$, and use that to approximate $k(0.001)$.

Near $x = 0$,

$$k(x) \approx k(0) + k'(0)(x - 0),$$

and $k(0) = (-1)^2(3) = 3$.

Therefore $k(0.001) \approx 3 + (-8)(0.001) = 2.992$.

(c) Find $m'(0)$ given $m(x) = \sin\left((f(x))^3\right)$.

$m'(x) = \cos\left((f(x))^3\right)(3\ (f(x))^2 f'(x))$,

so $m'(0) = \cos(27) \cdot (27)(4) = 108\cos(27)$.

3. (14 points) Find the values of the constants $a,\ b$ and c such that the function

$$f(x) = ax^2 + bx + c$$

"fits" the function

$$g(x) = -2\ln(x+1) + 0.5e^x + 1.5\sin(x)$$

near $x = 0$ in the sense that:

$$g(0) = f(0), \quad g'(0) = f'(0) \quad \text{and} \quad g''(0) = f''(0).$$

Show all work.

We have $g(0) = 0 + 0.5 + 0 = 1/2$ and $f(0) = c$,
so $\boxed{c = 0.5}$.

We find the first derivatives:

$$g'(x) = \frac{-2}{(x+1)} + 0.5e^x + 1.5\cos(x), \quad \text{and} \quad f'(x) = 2ax + b.$$

Thus, $g'(0) = -2 + 0.5 + 1.5 = 0, \quad$ and $\quad f'(x) = b$,

so $\boxed{b = 0}$.

Second derivatives give:

$$g''(x) = \frac{2}{(x+1)^2} + 0.5e^x - 1.5\sin(x) \quad \text{and} \quad f''(x) = 2a.$$

Thus,
$g''(0) = 2 + 0.5 - 0 = 2.5 \quad$ and $\quad f''(0) = 2a$,

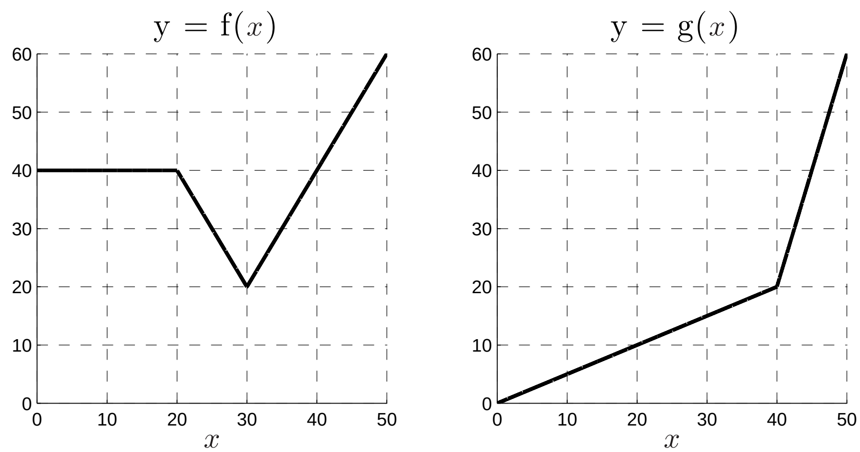
so $2a = 2.5$ which gives $\boxed{a = 1.25.}$

$a =$ 1.25

$b = 0$ 0

$c =$ 0.5

4. (10 points) Consider the graphs of $f(x)$ and $g(x)$ below. Let $h(x) = f(g(x))$.



(a) Evaluate $h'(30)$ exactly. Show your work.

$h'(x) = f'(g(x))g'(x)$.

At $x = 30$ we have $g(30) = 15$ and $g'(30) = 0.5$. Thus $h'(30) = f'(15)0.5$. However, $f'(15) = 0$, so

$$h'(30) = 0.$$

(b) Determine the range of values of x for which $h'(x) < 0$. Justify your answer.

Note, for $h'(x) < 0$ we need to be in the x range where $f'(x) < 0$, since $g'(x) > 0$ for all x.

We have $f'(x) < 0$ for $20 < x < 30$, so $g'(x)$ must be between 20 and 30. We have $g(40) = 20$ and the slope of g for $x > 40$ is 4. Thus, as $g(x)$ increases by 10, x increases by 2.5, so $g(42.5) = 30$.

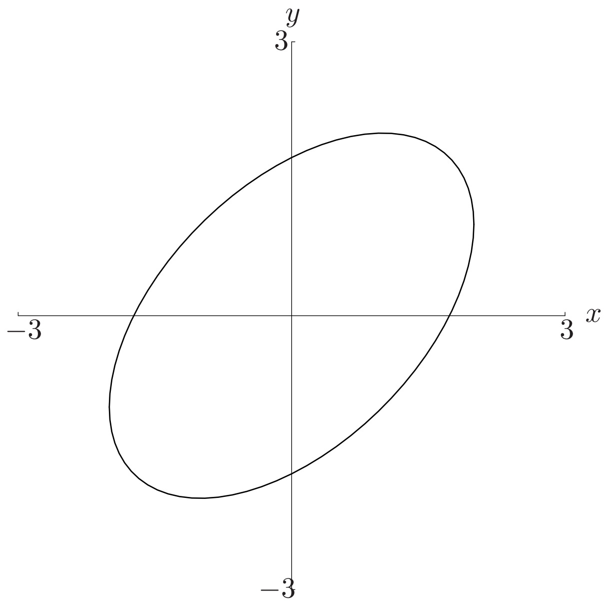
Thus, the range of values of x such that $h'(x) < 0$ is $\boxed{40 < x < 42.5}$.

Note that f is not differentiable at $x = 20$ or $x = 30$, so the inequality does not include the endpoints.

5. (12 points) The graph of

$$x^2 - xy + y^2 = 3$$

is a "tilted" ellipse (see the figure below). Among all points (x, y) on this graph, find the points that have the largest and smallest values of y. [Hint: Look at the figure to consider the conditions that would be true for y to take on largest or smallest values.] Be sure to show all work in order to justify your answer (*i.e.*. estimating points from a graph will not be sufficient).



Note that the largest and smallest values of y occur when $\frac{dy}{dx} = 0$. Taking the derivative of both sides with respect to x gives

$$2x - \left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} = 0,$$

so

$$(2y - x)\frac{dy}{dx} = y - 2x,$$

which gives

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}.$$

Therefore $\frac{dy}{dx} = 0$ when $x = \frac{y}{2}$.

Substituting into the original equation:

$$\frac{y^2}{4} - \frac{y^2}{2} + y^2 = 3; \quad y^2 = 4; \quad \text{so } y = \pm 2.$$

Solving for x when $y = \pm 2$ gives the points $(1, 2)$ and $(-1, -2)$.

Largest y value is associated with the point: (1,2)

Smallest y value is associated with the point: (-1,-2)

6. (5 points) Parasitoids are insects that lay eggs in, on, or close to other (host) insects. Their larvae then devour the host insect, resulting in the death of the host. The likelihood of escaping parasitism may depend on parasitoid density. One such model sets the probability, P, of escaping parasitism as:

$$P = f(D) = \left(1 + \frac{a\,D}{k}\right)^{-k}$$

where D is the parasitoid density and a and k are positive constants.

Determine whether the probability of escaping parasitism increases or decreases as parasitoid density increases. Justify your answer.

First we consider the derivative of P with respect to D:

$$P' = -a\,\left(1 + \frac{aD}{k}\right)^{-k-1}$$

which is negative, thus P decreases as D increases.

7. (12 points) Let $P = f(t)$ be the total amount, in trillions of barrels, of the world's reserves of petroleum in year t.

(a) What does the statement $f(2006) = 1.2$ tell you about the petroleum reserves?

In 2006 there were 1.2 trillion barrels in the world's petroleum reserves.

(b) Evaluate and interpret $f^{-1}(1.2)$.

This gives the year in which there was 1.2 trillion barrels of petroleum in the world's reserve and from above we have that was the year 2006.

(c) What does the statement $f'(2006) = -0.003675$ tell you about the petroleum reserves?

This tells us that in 2006 the world's reserves of petroleum was decreasing at the rate of 0.003675 trillion barrels per year. This means that in 2007 the world would have approximately 1.2-0.003675 = 1.196 trillion barrels of petroleum in its reserves.

(d) Evaluate and interpret $\left(f^{-1}\right)'(1.2)$.

This value can be found as

$$\left(f^{-1}\right)'(t) = \frac{1}{\left(f'(f^{-1}(t))\right)}.$$

Therefore $\left(f^{-1}\right)'(1.2)^{-1} = \left(f'(f^{-1}(1.2)\right) = -(0.003675)^{-1} = -272.11$ years per trillion barrels. This states that at the 2006 rate it would take approximately 272 years to use up a trillion barrels of the world's petroleum reserves. [Note that the rate of depletion is expected to increase greatly in the next few years—*i.e.*, f' becomes more negative—so this expectation is not reasonable.]

8. (17 points)

(a) A small company makes x hand-painted tiles daily at a cost of $C(x) = 125 + 30x + 2x^{3/2}$ dollars. What daily production level minimizes the *average* cost—*i.e.*, the cost per tile?

First define the average cost

$$A(x) = \frac{C(x)}{x} = \frac{125}{x} + 30 + 2x^{1/2}.$$

We find $A'(x)$ and any critical points:

$$A'(x) = -\frac{125}{x^2} + x^{-1/2}.$$

Thus A' is undefined if $x = 0$ (not in the domain of A), and $A'(x) = 0$ if $x = 25$.

Thus, we have one critical point at $x = 25$.

Note that

$$A''(x) = \frac{250}{x^3} - 0.5x^{-3/2},$$

and $A''(25) > 0$. Thus, $x = 25$ is a local min, and since this is the only critical point and the function is continuous on its domain, $x = 25$ is the absolute minimum.

(b) If the the company sells each tile for \$75, how many tiles should they make daily in order to maximize daily profit?

First define the profit function $\pi(x) = R(x) - C(x)$, ie

$$\pi(x) = 75x - 125 - 30x - 2x^{3/2} = -125 + 45x - 2x^{3/2}.$$

The critical points occur when $\pi'(x) = 0$ or is undefined. For x in the domain, the only critical point is when $\pi(x) = 0$, or

$$\pi'(x) = 45 - 3x^{1/2} = 0, \text{ giving } x = 225.$$

Once again we must test the critical point. The second derivative here is

$$\pi''(x) = -\frac{3}{2}x^{-1/2}$$

which is negative for $x > 0$, so the critical point gives a local max. Since it is the only critical point on the domain and the function is continuous, the maximum profit occurs when the company makes 225 tiles per day.